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Polarisation and Poverty Reduction in Africa: The Devil is in the Choice of Equivalence Relation

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Abstract

The focus of this paper is the low observed mean consumption elasticity of poverty in Africa, and the suggestion that polarisation of national distributions, specifically the non-parametric 'relative distribution' method, is essential to understanding the low regional elasticity. The version of the methodology adopted results in a measure of absolute polarisation. We show that the results obtained for 24 countries in the region are entirely a product of this choice, and while preference for translation-invariance is a normative matter, claims regarding changes in distributions are not. There is no evidence of distributional changes unaccounted for by standard measures of inequality and mean consumption. Which, in turn explain the evolution of poverty levels in the 24-country sample. Given that changes in mean consumption and inequality, among the sample countries, account for both the changes in the chosen measure of polarisation and the evolution of poverty, there is no distinct role for the chosen measure of polarisation in accounting for the evolution of poverty in the region.

Keywords Polarisation, Sub-Saharan Africa, Log Normal, Relative Distribution, Translation-Invariance

JEL Classification C14, C46, D63, O15

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1 Introduction

In this paper we consider Clementi, Fabiani, and Molini (2019), hereafter CFM, in which they set out to explain the "low growth-to-poverty elasticity characterising Africa" (p.208). The starting point for CFM is the observation in Beegle et al. (2016) that the regional poverty ratio for Sub-Saharan Africa (SSA) reduced by only 13 percentage points between 1990 and 2012 despite sustained regional growth.[1] CFM consider two possible explanations for the low growth elasticity of poverty: that GDP growth is overstated, or that an increase in inequality may have "limited the pro-poor content of the growth" (p.410). The first is recognised as a potential partial explanation as household consumption growth was indeed slower than GDP growth for SSA over the period; however, since growth in household consumption was still robust at 2.32% per year, a further explanation is still required. The possibility that increased inequality may be the explanation is also dismissed citing several studies, including Thorbecke (2013), Beegle et al. (2016), and Cornia (2017), showing that there is no clear trend in inequality in SSA countries, with as many showing declines as increases. CFM then argue that while there is no clear trend in 'standard inequality measures' there are significant, and consistent, distributional changes inhibiting the translation of growth into poverty-reduction. To identify and analyse these distributional changes CFM employ the non-parametric 'relative distribution' approach to estimating polarisation.

Polarisation is a notion increasingly used to analyse income and consumption distributions. Three distinct strands of the literature emerged at approximately the same time, one centred on Foster and Wolfson (1992), henceforth FW, and Wolfson (1994), the second initiated by Esteban and Ray (1994), henceforth ER, and the third following Morris, Bernhardt, and Handcock (1994). Both FW and ER identified the key components of polarisation as increased 'spread' or heterogeneity between distinct subgroups, and reduced disparity or increased homogeneity within subgroups, FW restricted their analysis to bipolarisation, whereas ER allowed for a "small number of significantly sized groups". From these 'axioms', FW and ER develop summary measures of polarisation, P^{FW} and P^{ER},

[1] The figures quoted are based on the recognised international poverty line of USD 1.9 (in 2011 PPP) per day, the reduction being from 56% to 43%.

respectively. The relative distribution method, introduced by Morris, Bernhardt, and Handcock (1994), and developed by Handcock and Morris (1998, 1999), is a non-parametric approach allowing comparisons along the entire distribution. To compare the initial 'reference' distribution with the 'comparison' distribution it is first necessary to match the location of the two distributions. To do this, a location adjustment is applied to the reference distribution transforming it into the counterfactual reference distribution; this adjustment can be to match the mean or the median, and effected by additive or multiplicative transformation, depending on 'the nature of the data'. The relative density of the distributions can then be compared at various quantile intervals; commonly, relative decile densities are used. The advantage of this method, as CFM state (p.410), is that it allows a detailed analysis of changes at specific parts of the distribution. These relative decile densities can then be used to form a summary measure, the Median Relative Polarisation (MRP). It is this relative density approach, and the MRP thus derived, that CFM claim is essential to the understanding of the low growth elasticity of poverty, since it allows the identification of "[d]istributional changes that went undetected by standard inequality measures" (p.408). More specifically they identify that in 'almost all' of the twenty-four countries analysed there was an increase in polarisation, caused by a 'hollowing out' of the middle and a 'concentration' in the lowest and highest deciles of the distribution.

There are two issues with this analysis: first, the notion that the evolution of poverty headcount is unaccounted for by changes in mean consumption and inequality, and second, the claim that the results in CFM constitute evidence of distributional changes unaccounted for by changes in mean consumption and inequality. On the first issue, taking the Bourguignon (2003) 'identity' model, which expresses poverty headcount as a function of mean income and inequality, and applying it to the start-date, and end-date, cross-sections of the twenty-four countries covered in the CFM survey provides a good fit to the data, with an R-squared of 0.99 in both cases.[2] In these countries, at least, there are no changes unaccounted for. The broader problem with the regional elasticity conundrum is that the evidence cited confuses the regional elasticity with national inequality. The regional

[2] The 'identity' model assumes log Normality. The underlying assumption that national income distributions closely approximate log Normality has been confirmed in several studies; see for example Lopez and Servén (2006), Kalwij and Verschoor (2007) and Klasen and Misselhorn (2008). If anything, national consumption distributions fit log Normality even more closely; see Battistin, Blundell, and Lewbel (2009).

elasticity describes how regional poverty is affected by regional mean consumption and regional inequality. However, while regional mean consumption and regional poverty are entirely determined by national mean consumption and poverty, regional inequality does not supervene on national inequality in this way, it is also affected by the dispersion between country mean consumption levels.

The second issue is essentially one of interpreting results obtained using an absolute measure as if they were obtained using a relative measure. CFM follow an adaptation of MRP that was introduced by Clementi and Schettino (2015), whereby additive median adjustments (AMA), rather than multiplicative median adjustments (MMA), are applied to consumption data to form the counterfactual reference distribution, thus creating a measure of absolute polarisation, as they acknowledge (Clementi and Schettino, 2015, p.938). It is the application of AMA to consumption data, as opposed to log-consumption data, that causes the Clementi and Schettino (2015) measure to be distinct from the 'relative distribution' method as outlined by Handcock and Morris.[3] For this reason, to distinguish the two procedures we will refer to MRP with AMA applied to income/consumption data as MRP+. This same technique is adopted by Nissanov and Pittau (2016), Clementi, Dabalen, Molini, and Schettino (2017), Clementi, Molini, and Schettino (2018), hereafter CMS, and the paper of interest here, CFM, we will refer to this collectively as the MRP+ literature.

That the construction of the counterfactual reference distribution does not preserve relative inequality is explicitly recognised (CFM, Footnote 3, p.412), and it is rightly noted that the preference for scale-invariant/relative measures or translation-invariant/absolute measures is an open normative question. However, whether there are changes in a distribution unaccounted for "standard inequality measures" and changes in mean consumption (Claim 1) is not a normative question; it is a positive question about the state of the world. Since MRP+ is derived using AMA, which does not preserve "standard measures of inequality", MRP+=0 cannot be the appropriate null hypothesis for testing Claim 1. In case of an increase in mean consumption of 50%, with inequality unchanged at Gini=0.4, henceforth Scenario 1,

[3] This may look superficially like the technique used in Handcock and Morris (1998); however, there the AMA are applied "because the data units are differences in log wages" (p.67), which as they state is equivalent to applying MMA to income data.

the scale-invariant advocate would say inequality was unchanged, and the translation-invariant advocate would say that absolute Gini had increased 50%, but there would be no disagreement about the state of the world, so these results are easily reconciled. Some measures of polarisation, for example P^{FW}, have a related property called the 'compromise' property (see Chakravarty, 2009, p.106), where the absolute polarisation is obtained by multiplying the relative measure by the median. Obviously, there can be no such property in the case of MRP, and its absolute counterpart, MRP+, not least since their range is bounded on the interval [-1, +1].

So, we perform a calibration exercise by calculating MRP and MRP+ for a change in distribution where we know that there are no changes unaccounted for by changes in mean consumption and inequality. We take scenario 1 with both distributions being log Normal. P^{FW} and P^{ER} are of course unchanged as they are scale-invariant and calculating MRP (i.e., using MMA) yields an index of zero, as expected. However, using AMA to create the counterfactual reference distribution, a process described in section 3, results in an MRP+ of +0.288. Since the whole distribution is completely specified by the mean consumption and Gini index, we know *ex ante* that claims like Claim 1 are false. However, the MRP+ literature would dismiss their nominated null hypothesis of MRP+=0. Now we can all agree that there is a change in MRP+; this, however, tells us nothing about Claim 1. The appropriate null hypothesis against which to test Claim 1, given an increase of 50% and unchanged Gini=0.40, would be MRP+=+0.288.[4] Nor, as we shall see, can the 'shape effect' graphs support Claim 2 (a) the 'hollowing out of the middle', or 2 (b) the 'concentration' in the highest and lowest deciles, collectively referred to as Claim 2, without first undertaking the calibration exercise.

We then use the same approach, with mean consumption and inequality data for the CFM sample, to calibrate the appropriate null hypotheses for testing Claim 1 and Claim 2. This is not to replicate the process in CFM, as we do not have their data; however, if their empirical results match the parametrically derived null hypotheses, we can conclude that these results are not driven by changes in distributions undetected by 'standard' measures of inequality, but by their choice of invariance condition. Likewise, if the empirically derived

[4] Here, we have assumed log Normality, but the conclusion is not dependent on it, robustness checks using the log-Uniform (reciprocal), and log-Logistic, distributions yield MRP+ results of +0.342, and +0.273, respectively.

shape effect histograms match those derived parametrically then there is no support for CFM's conclusions regarding specific distributional changes.

The paper will proceed as follows: In section 2, we will introduce four notions of polarisation, those of FW, and of ER, and then MRP, and MRP+. In section 3, we will apply these measures to a hypothetical distribution to compare the results in terms of the summary measure. We will also compare the 'shape effect' graphs derived under MMA and under AMA. Section 4 will focus first on the results of CMS, looking at Ghana in detail, before addressing the data in CFM, again deriving the adjusted null hypotheses considering the metric construction. Section 5 concludes.

2 Polarisation

Increased interest in the concept of polarisation following ER, FW, and Wolfson (1994) led Duclos, Esteban, and Ray (2004, p.1737) to declare that polarisation was now widely accepted as being a phenomenon distinct from inequality. In this section we will first introduce the principal streams of the polarisation literature, those initiated by FW, by ER, and by Morris, Bernhardt, and Handcock (1994), and then discuss the specific form of the 'relative distribution' method initiated by Clementi and Schettino (2015) and employed by CMS and CFM.

2.1 Foster and Wolfson

The stream of the polarisation literature initiated by FW and Wolfson (1994) was directly motivated by the growing concern over the notion of a 'disappearing middle', as indicated by the title of FW, and the first sentence of Wolfson (1994): "A significant innovation in the discussions of income inequality is the addition, since the early 1980's of the "disappearing middle class"…" (p.353). FW pointed to the fact that, while many studies had identified a reduction in the density of the 'middle', the very notion of the middle was arbitrary, so what they proposed was to create a "range-free approach to measuring the middle class and polarisation" (p.247). FW focus only on the case of polarisation involving two groups, bipolarisation. The formalisation of polarisation in FW is characterised by two contributory notions: 'increased spread' and 'increased bipolarity'. The increased spread component says that one distribution has higher polarisation when "no matter what the range of families is chosen around the median family, the range of incomes (or “spread”) necessary to capture all the families is larger” (FW, p.249). The increased bipolarity component says that polarisation is higher when the “average distance from the median income … is higher for every range of families about the median” (ibid.).

2.2 Esteban and Ray

The stream initiated by ER is based on the ‘identification-alienation framework’, the notion that the alienation between ‘clusters’ within a society will increase with the level of difference between clusters but will also be exacerbated by the level of similarity within clusters. The motivation for their approach was the belief that in this form, high levels of polarisation may be the precursor to social tension, even conflict (ER, p.820). The only restriction on the number of ‘clusters’ is that there should be a “small number of significantly sized groups” (ER, p.824), so this approach is distinct from pure bi-polarisation in that sense, but the notion that polarisation is identified by a “high degree of homogeneity *within* each group” and a “high degree of heterogeneity *across* groups” (ER, p.824, emphasis as in the original) is like the FW framework.

While the two approaches had different motivations, the frameworks shared the intuition that polarisation increases when the gaps between groups increase, but also increases when the gaps within groups are reduced. Both streams emphasised that while these two properties were intimately related to inequality, they related to it in opposite senses. The fact that progressive transfers within groups would simultaneously lower inequality and raise polarisation was given as evidence of the distinctness of the two concepts (see FW, p.251-2, and ER, p.825). The common ground between the streams was also made evident when Esteban, Gradín, and Ray (2007), henceforth EGR, extended the ER measure to a more general framework and demonstrated that a form of the FW measure can be derived as a special case.

2.3 Morris, Bernhardt, and Handcock

Meanwhile a separate, and quite distinct, approach to estimating polarisation was underway, starting with Morris, Bernhardt, and Handcock (1994), and subsequently developed by Handcock and Morris (1998, 1999). Here the motivation was not to establish a notion separate from inequality, but to identify whether the polarisation of jobs (growth in high-wage and low-wage jobs at the expense of the middle) was behind an increase in income inequality. They develop a non-parametric approach which involves the comparison of the relative population density of two distributions. It is this strand of the polarisation literature that CFM follow. The relative density of the comparison distribution Y to the reference distribution Y_0 is defined as:

$$g(r) = \frac{f(y_r)}{f_0(y_r)} \tag{1}$$

where $f(.)$ and $f_0(.)$ are the density functions of Y and Y_0 respectively, and $y_r = F_0^{-1}(r)$ is the quantile function of Y_0.[5]

As Handcock and Morris (1998, 1999) state, this relative distribution can then be decomposed into 'location' and 'shape' components. This involves creating a third counterfactual/adjusted distribution Y_A which retains the shape of Y_0 but is adjusted to the location of Y.[6] Then the decomposition is as follows:

$$\frac{f(y_r)}{f_0(y_r)} = \frac{f_A(y_r)}{f_0(y_r)} \times \frac{f(y_r)}{f_A(y_r)} \tag{2}$$

the first component on the right-hand side representing the 'location effect' and the second component the 'shape effect'.[7] Of crucial importance is the form of the transformation of the reference distribution, i.e., $f_A(y_r)$. This can involve matching the mean or the median, by employing additive or multiplicative adjustments to the reference distribution. All four combinations have been used in the empirical literature, as will be discussed below, but here we simply note that CFM, consistent with the rest of the MRP+ literature, elect for AMA. So, in CFM the counterfactual distribution is formed by adding ρ to the reference distribution, where ρ is the median of the comparison distribution minus the median of the reference distribution. Decile thresholds/cut points of the counterfactual distribution are then identified, and the proportion of the comparison distribution divided by the proportion of the counterfactual reference distribution between each pair of cut points is estimated.

[5] This is equation (2.3) in Handcock and Morris (1999, p.22), and equation (1) in CFM (p.412).
[6] In the original formulation Morris, Bernhardt, and Handcock (1994) made the location adjustment to the comparison distribution, deflating it to the reference location, but CFM, in keeping with Handcock and Morris (1998), adjust the reference distribution.
[7] This is equation (3.1) in Handcock and Morris (1999, p.45), and equation (2) in CFM (p.412).

The resulting distribution, $g^t(i)$, is the location-matched relative distribution, or 'shape effect'.[8]

Whichever adjustment mechanism is used, from these resulting relative decile densities, a summary measure, the MRP is calculated as follows:

$$MRP = \frac{4}{Q-2}\sum_{i=1}^{Q}\left|\frac{i-\frac{1}{2}}{Q}-\frac{1}{2}\right| g^t(i) - \frac{Q}{Q-2} \tag{3}$$

where $Q/_{(Q-2)}$ is a renormalisation factor.[9] CFM also follow Handcock and Morris is favouring deciles for representing the 'shape effect' and calculating the MRP, so we substitute Q=10.

2.4 The case for additive adjustments

The MRP+ literature consistently advocates the use of AMA in their studies, citing both normative and instrumental reasons. CFM, for example, cite Kolm (1976) and Atkinson and Brandolini (2010) in support of the normative argument, and claim that absolute inequality better reflects people's concern about the "widening economic divide", citing the experimental results of Amiel and Cowell (1999), but it is the discussion of shape preservation that is central to their case. CFM state the case as follows: "this approach appears well-suited to a counterfactual density decomposition, since the visual impact of equal additions is a sliding of the reference density along the x-axis with no change in shape …. In contrast, a *multiplicative* location shift has the drawback of affecting the shape of the reference distribution" (Footnote 3, p. 412, emphasis in the original). This is consistent throughout the MRP+ literature; see Clementi and Schettino (2015, Footnote 3, p.933), Clementi, Dabalen, Molini, and Schettino (2017, Footnote 3, p. 613), and CMS (Footnote 10, p. 278).[10] From this position the MRP+ literature elects for AMA and hence constructs a

[8] This is the equivalent, accounting for the reversal of the adjustment process, to Morris, Bernhardt, and Handcock (1994, p.217), where $g^t(i)$ is identified as the "proportion of year t's earners whose median adjusted incomes fall between each pair of the quantile cut points, divided by the proportion of the baseline year".

[9] Equation (3) also appears as equation (7) in CMS (p.279). The expression includes a re-normalisation factor of $Q/_{(Q-2)}$. The arbitrary nature of this adjustment will be discussed in Appendix A, but all that is required here is that we replicate the respective methods used.

[10] This, of course, strays from the normative into the positive, and as a matter of historical observation it is the case that distributional changes tend to preserve the shape in log-income/expenditure terms, not in

measure of absolute polarisation, as acknowledged by Clementi and Schettino (2015, p.938) when comparing their methodology (MRP+) to other measures of polarisation: "To avoid biased comparison among different measures of polarisation, in line with the approach used earlier … we construct an 'absolute' counterpart to the Foster-Wolfson index by multiplying it by the median."[11]

Both additive and multiplicative adjustment mechanisms are employed in the empirical literature, so we return to Handcock, and Morris (1998, 1999). Since the location adjustment mechanism is so important to the interpretation of the results, we will quote in full the paragraph in Handcock and Morris (1998) that deals with the issue:

> "While the relative distribution is scale-invariant, the decomposition developed below is not. This is because the concept of location shift is inherently scale dependent: A multiplicative shift on the original scale is an additive shift on the log scale. The appropriate scale is driven by the specific application, and the analyst should choose the scale according to the nature of the data. In the discussion below, we use an additive median location shift. We choose the median because population quantiles are a natural, robust, and scale invariant unit of measurement, and an additive shift because the data units are differences in log wages." (p.67).

There are four possibilities for the location shift, choosing between additive and multiplicative, and choosing between mean (C) and median (M), described here with respect to Y and Y_0 being in linear income/expenditure terms.

(i) $f_A(y_r) = f_0\big(y_r.(C_t/C_0)\big)$

(ii) $f_A(y_r) = f_0\big(y_r.(M_t/M_0)\big)$

(iii) $f_A(y_r) = f_0\big(y_r + (C_t - C_0)\big)$

(iv) $f_A(y_r) = f_0\big(y_r + (M_t - M_0)\big)$

All four versions have been used in the empirical literature: (ii) is used by Alderson, Beckfield, and Nielsen (2005, p.410), (iii) is advocated in Jenkins and van Kerm (2005), and

income/expenditure terms, otherwise the variance of income levels would not be an order of magnitude greater than the variance in relative inequality.

[11] See also CFM (Footnote 3, p.412), where they explicitly acknowledge that AMA is consistent with the absolute inequality concept, while MMA is consistent with the more widely applied relative inequality concept. The same point is also made by Jenkins and van Kerm (2005, pp.50-1).

(iv) is the version favoured by Clementi and Schettino (2015) and deployed in the subsequent MRP+ literature. The method used in Handcock and Morris (1998) looks superficially similar to (iv); however, noting that they are dealing with log-income data, we take logs of (iv) which yields (i).[12] Indeed in the following paper Handcock and Morris (1999) are specific: "In our application, for example, we would obtain the same relative distribution from the ratio of earnings as we do from the difference in log-earnings" (Handcock, and Morris, 1999, p. 26).

When income/expenditure data are being considered, as opposed to log-income/expenditure, additive adjustments will not preserve 'standard [relative] measures of inequality', as acknowledged by CFM (Footnote 3, p.412).[13] While this is in keeping with the normative preference referred to in CFM, it means that drawing conclusions about 'changes in the distribution undetected by standard measures of inequality' cannot be made directly from the data; it is first necessary to calibrate the expected results given an unchanged standard measure of inequality. The earlier quotation from Handcock and Morris (1999) would seem to suggest that MMA would be preferred for income/expenditure distributions, and AMA for log-income/expenditure distributions. Nevertheless, we accept that the preference between equivalence relations is an open normative question. However, as CFM state (p.412) the choice of absolute inequality concepts has "far-reaching implications"; it is essential to keep a clear track of the consequences as these concepts are deployed, otherwise intuition may easily be confounded. We will now look at the implication of CFM's choice in sections 3 and 4.

3. Calibration, and interpretation, of MRP and MRP+

3.1 Calibration

This section is a calibration exercise. We will examine a scenario in which we know there are no changes in distribution unaccounted for by changes in mean consumption and inequality, i.e., in which Claim 1 is false. We take a log Normal distribution with initial mean consumption of 100, final mean consumption of 150, and with constant inequality of

[12] For distributions symmetrical in expenditure, the median equals the mean; for distributions symmetrical in log-expenditure, such as log Normal, log (Median expenditure) equals Mean (log-expenditure).
[13] Additionally, the adjustment mechanism (iv) is not mean preserving.

Gini=0.40. The log Normal distribution is a two-parameter distribution; once the mean and standard deviation of log consumption are known the whole distribution is known, and these parameters are in turn a function of mean consumption and the Gini index. We will start with a central assumption of log Normal consumption distributions but will perform robustness checks employing log Logistic and log Uniform (Reciprocal) distributions.[14]

For a log Normal distribution, the proportion of the population, P, with consumption below some threshold T is given by:

$$P = \Phi\left(\frac{ln(T/C)}{\sigma} + \frac{\sigma}{2}\right) \tag{4}$$

where Φ is the cumulative standard Normal distribution, C is mean consumption, and σ is the standard deviation of log consumption given by the expression $\sigma = \sqrt{2}\,\Phi^{-1}\left(\frac{Gini+1}{2}\right)$, where $\Phi^{-1}(.)$ is the inverse of the cumulative standard Normal distribution.[15]

We start with three inputs: G_0, G_t, and CG; respectively the inequality, as measured by the Gini index, for the start date and end date, and the cumulative growth between the start and end date in %. The standard deviation of log-consumption, σ is then calculated as follows:

$$\sigma_0 = \sqrt{2}.\Phi^{-1}\left(\frac{G_0 + 1}{2}\right) \tag{5}$$

and

$$\sigma_t = \sqrt{2}.\Phi^{-1}\left(\frac{G_t + 1}{2}\right) \tag{6}$$

where $\Phi^{-1}(.)$ is the inverse of the cumulative standard Normal distribution.

From the respective standard deviations (eqs. (5) and (6)) and mean expenditure levels (C_0 and C_t, where $C_t = C_0.(1 + \frac{CG}{100})$) we can calculate the median consumption level at the start and end date which, on the assumption of log Normality, are given by:

$$M_0 = C_0.e^{-\left({\sigma_0}^2/2\right)} \tag{7}$$

[14] The Logistic and Uniform distributions have excess kurtosis of plus 1.2 and minus 1.2 respectively, so they cover a broad range of distributions centred on the Normal.

[15] As used by Bourguignon (2003) with the poverty threshold Z in place of T to express the poverty ratio as a function of mean income and the Gini index.

and

$$M_t = C_t . e^{-\left(\sigma_t{}^2/_2\right)} \tag{8}$$

To calculate MRP we first take the mean consumption and Gini index for the start date and calculate the theoretical decile consumption thresholds.[16] Given that the distribution is log Normal the quantile function is simply the inverse of equation (4), the cumulative density function:

$$T = C . e^{\sigma\left(\left(\Phi^{-1}(P)\right)-\frac{\sigma}{2}\right)} \tag{9}$$

So, the decile thresholds are given by:

$$DT_i = C . e^{\sigma_o\left(\left(\Phi^{-1}\left(\frac{i}{10}\right)\right)-\frac{\sigma_o}{2}\right)} \tag{10}$$

where $i \in (1,2,\ldots,9)$, and $DT_{i,0}$ indicates the threshold, at the start date, separating decile i from decile $i + 1$.

The parametric decile thresholds are then multiplied by the ratio of the end point median (M_t) to the starting median (M_o) to create the parametric 'reference distribution' thresholds, RT_i:

$$RT_i = DT_i . \left(\frac{M_t}{M_o}\right) \tag{11}$$

The resulting thresholds, or decile 'cuts points', are displayed on Figure 1, represented by squares and labelled 'MMA reference distribution'.

The equivalent thresholds using the AMA are:

$$RT_i = DT_i + (M_t - M_o) \tag{12}$$

These resulting thresholds are also displayed on Figure 1, represented by triangles, and labelled 'AMA reference distribution'.

[16] We reserve Greek lower case for parameters of the log-consumption distribution, and Roman capitals for parameters of the consumption distribution.

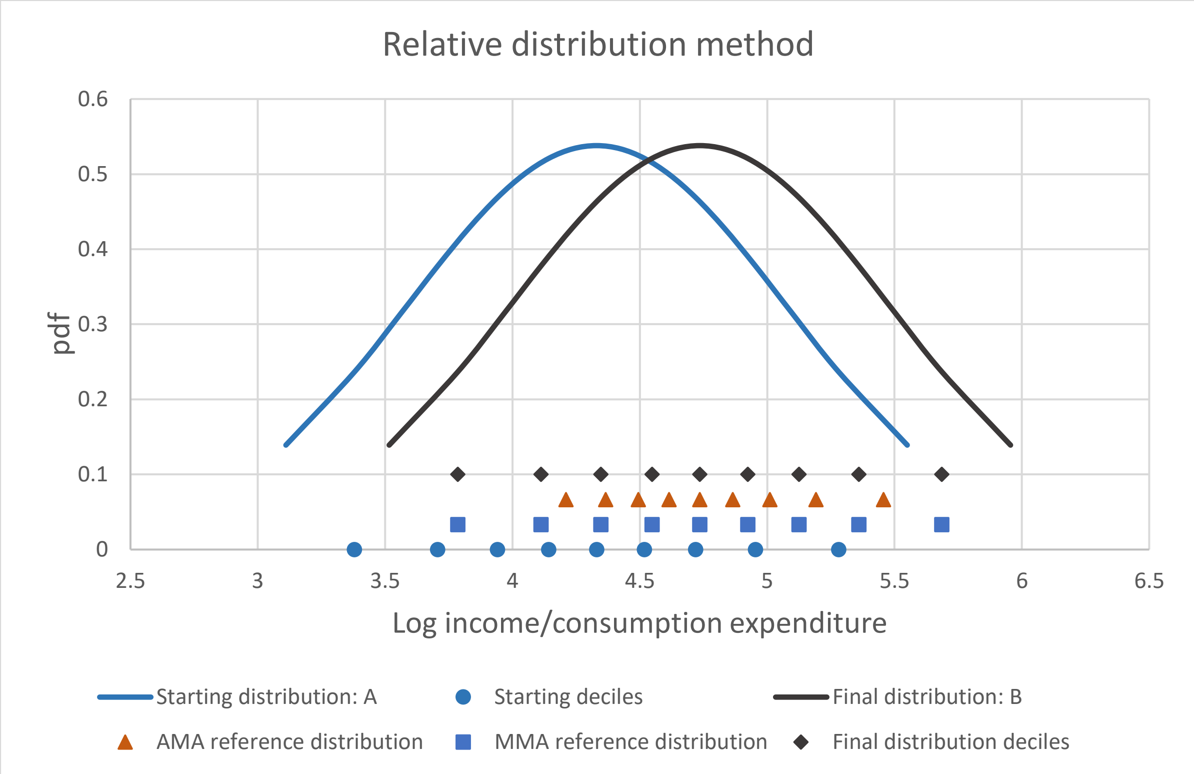

Figure 1: The effect on the 'reference distribution' thresholds, under MMA and AMA, for an increase of 50% in mean income/consumption expenditure, unchanged Gini=0.40, and a log Normal distribution.

Then, taking the mean consumption and Gini for the end date, we derive a relative density for the closing distribution relative to the 'reference distribution'.

For a log Normal distribution, the proportion of the population, $P_{i,t}$, with consumption below RT_i, is given by:

$$P_{i,t} = \Phi\left(\left(\frac{ln(RT_i/C_t)}{\sigma_t}\right) + \left(\frac{\sigma_t}{2}\right)\right) \tag{13}$$

for $i \in (1,2, \dots ,9)$, and setting $P_{0,t} = 0$ and $P_{10,t} = 1$.

From this we can calculate the proportion of the population at the end date within each band of the reference distribution:

$$p_{i,t} = P_{i,t} - P_{i-1,t} \tag{14}$$

Now the relative density in each of the deciles of the reference distribution is:

$$RD_i = \frac{p_{i,t}}{p_{i,0}} = \frac{p_{i,t}}{0.1} = 10. p_{i,t} \tag{15}$$

These RD_is are the $g^t(i)$s used as inputs in the calculation of the MRP, see equation (3), and tabulating the RD_is produces the 'shape effect' graph, see Figure 2.

The relative distribution approach to polarisation with MMA leads to an MRP of zero. The relative distribution deciles are illustrated in Figure 2(a). However, using the relative distribution method with the AMA mechanism yields the relative distribution histogram shown in Figure 2(b).[17] This occurs because the AMA mechanism causes the reference decile thresholds to concertina inward, as can be seen in Figure 1. With mean consumption of 100, and Gini of 0.40, a log Normal distribution has a median consumption of 75.96, whereas a mean consumption of 150, and a Gini of 0.40 implies a median of 113.94, 50% higher. So, AMA adds 37.98 (113.94-75.96) to each quantile threshold. This raises the threshold of the bottom decile from 29.36 to 67.34; however, the threshold of the top decile rises by the same absolute amount, from 196.49 to 234.47. So, the first consumption level above the bottom decile would need to rise by 129% to avoid 'relegation', whereas the first consumption level below the top threshold would only have to rise by 19% to achieve 'promotion'.[18]

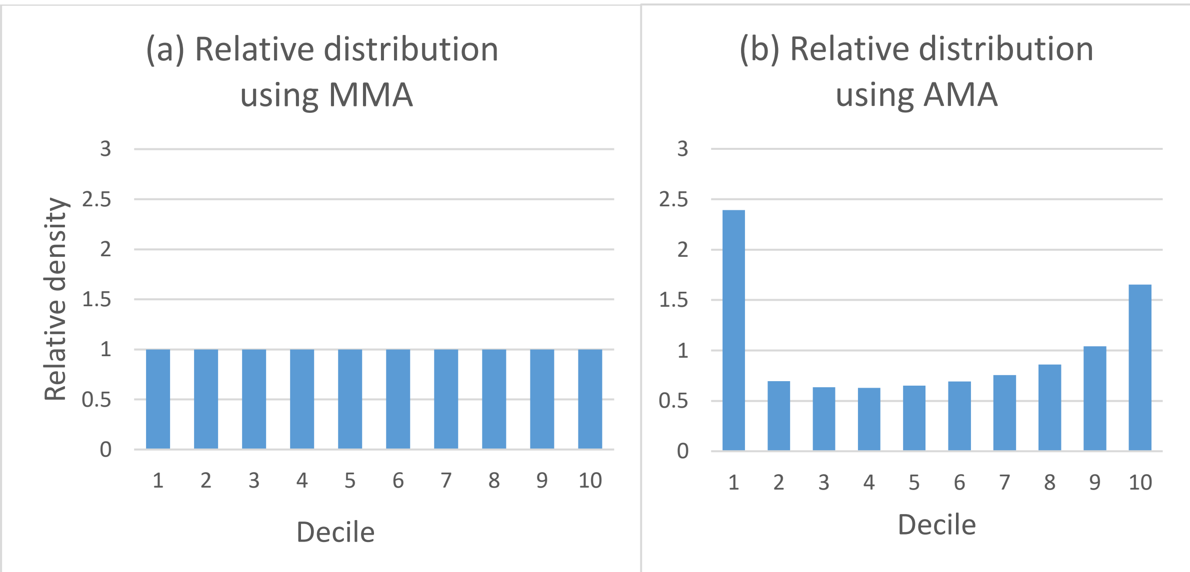

***Figure 2**: The decile relative density histograms for an increase in mean consumption expenditure of 50%, unchanged Gini=0.40, and a log Normal distribution; (a) using MMA, and (b) using AMA.*

[17] The general shape of the relative density histogram in Figure 2(b) will be familiar to followers of the MRP+ literature, see for example Clementi and Schettino (2015, Figure 1(c), p.935), Clementi, Dabalen, Molini, and Schettino (2017, Figure 2(c), p.619), CMS (Figure 1(d), p.281), and CFM (Figure 4(c), p.419).

[18] See Appendix B.1 for details of the derivation of the Decile relative densities.

For this hypothetical scenario, in which we know *ex ante* that Claim 1 is false, calculating MRP (i.e., using MMA) results in an index of zero, derived from the decile relative densities displayed in Figure 2(a). However, calculating MRP+ (i.e., with the counterfactual reference distribution formed using AMA) yields a result of +0.288, derived from the decile relative densities displayed in Figure 2(b). Of course, it is possible that national distributions may not conform to log Normality, so we perform a robustness check by constructing the relative decile density histograms, under the same change in parameters, but with the assumption of log Logistic, and log Uniform (Reciprocal), distributions. The results are displayed in Figures 3(a), and 3(b). From these decile data we calculate the MRP+ as +0.273, and +0.342 respectively.

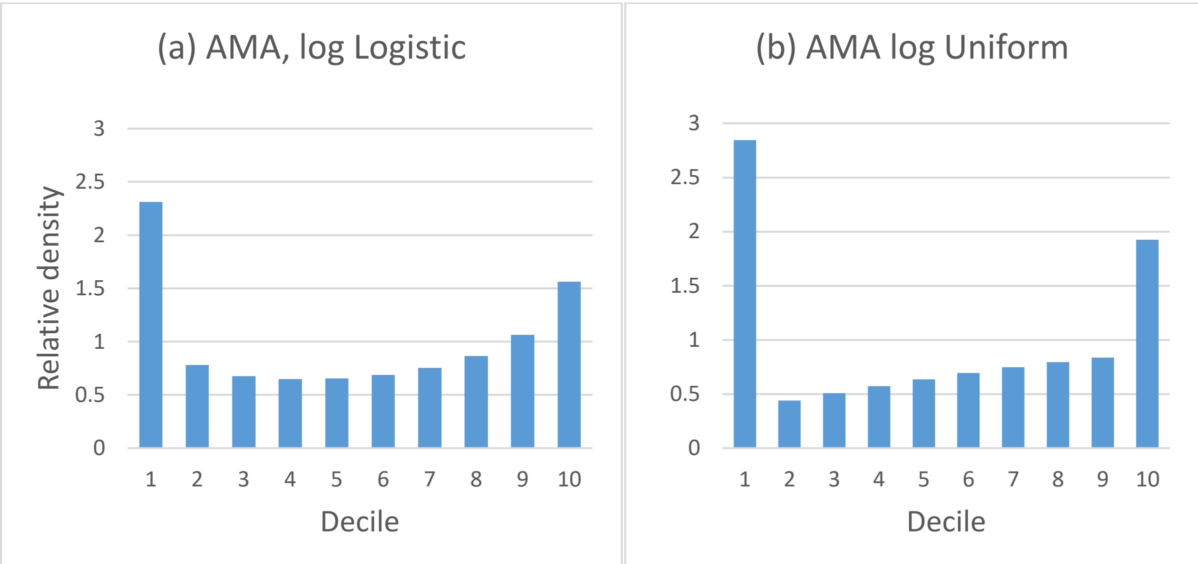

Figure 3*: The decile relative density histograms for an increase in mean consumption expenditure of 50%, unchanged Gini=0.40, using AMA under the assumption of (a) a log Logistic distribution, and (b) a log Uniform (reciprocal) distribution.*

3.2 Interpretation

The results of calculating the four measures for our hypothetical scenario, P^{FW}, and P^{EGR} unchanged, MRP=0.0, and MRP+=+0.288, are all derived from the same state of the world. Since the underlying transition between distributions was identical the conclusions drawn from the measures had better agree, as the choice of metric does not alter the state of the world.

In our hypothetical scenario the initial and final distributions are both entirely determined by the level of mean consumption and inequality, hence the transition from one distribution to the other is determined by the transition in mean consumption and Gini. Therefore, the results obtained in the calibration are the appropriate null hypotheses for claim 1. In the case of MRP+ the ‘shape effect’, and the resulting estimate of MRP+=+0.288, cannot be taken as evidence of ‘distributional change unaccounted for by changes in mean expenditure and inequality’.

This is not to claim that an MRP+ of +0.288 is evidence, by itself, of the log Normality of the distribution; for example, a mirror image of the relative decile densities in Figure 2(b) would produce the same MRP+, and in this case one, or both, of the distributions would be far from log Normal. However, if the entire ‘shape effect’ matches Figure 2(b) this is evidence of log Normality, since the cumulative density at eight points spread through the distribution would have to match, and inspection of Figures 3(a), and 3(b), will show that the relative decile distributions have distinct shapes. We would stress that we do not depend on this additional claim; for the central argument of this paper, it suffices that in this scenario MRP+=+0.288 is compatible with ‘no change in distribution unaccounted for by changes in mean expenditure and Gini’. This is also not to claim that changes in polarisation cannot explain changes in the distribution that may be undetected by standard measures of inequality, if the output in Figure 2(b) had appeared in 2(a), i.e., resulted from MRP, using MMA, then it would indeed be evidence of the preponderance of ‘downgrading’. The reduced density in the centre would show up as an increase in inequality, but the asymmetry would not be revealed by standard measures of inequality.

When mean expenditure rises by 50% and Gini remains 0.40 then in the case of P^{FW}, P^{EGR}, the appropriate test of Claim 1 is whether the closing value is statistically significantly different from the opening value, and for MRP the appropriate test is whether the estimated value is statistically significantly different from 0. However, in the case of MRP+, where AMA is employed, the appropriate test is whether MRP+ is statistically different from +0.288. If there were reason to believe that the distribution was closer to log Logistic, Battistin, Blundell, and Lewbel (2009) notwithstanding, then it could be argued that the null hypothesis should be an MRP+ of +0.274, but under no circumstances could the null hypothesis of MRP+=0 be justified for testing Claim 1.

With regards to claim2, there are many examples in the MRP+ literature of ‘shape effect’ histograms being cited as evidence of a ‘hollowing out of the middle’ and a ‘concentration’ in the top and bottom deciles of the expenditure distribution; see for example Clementi and

Schettino (2015, p.936), Clementi, Dabalen, Molini, and Schettino (2017, p.619-620), CMS (p.275), and CFM (p.418). However, recall that the decile relative densities represent the distribution relative to the counterfactual decile cuts derived using AMA; the 'shape effect' does not represent a change in the state of the world. In our hypothetical scenario, for example, there is no change in the shape of the distribution in log-expenditure space, so a scale-invariance advocate would simply say that there has been no change in the shape of the distribution. What about the translation-invariance advocate? Observing the transition in expenditure space, as opposed to log-expenditure space, it is certainly possible to construe a semantic sense in which the 'middle' of the distribution is less dense. The change in distribution in expenditure space is illustrated in Figure 4; this is the 'flattening' of the distribution discussed by Jenkins and van Kerm (2005) and cited by CFM (Footnote 3, p.412). However, if 'thinning middle' means reduced density in expenditure space, then the thinning is even throughout the distribution; as indicated on the figure, the density at the mean, at half the mean, and at twice the mean, are all reduced by one third. And if 'thinning' means increased distance between neighbouring points in the distribution, then again it can be said that the middle of the distribution is thinning, but the thinning is proportional to initial levels, so the thinning is greatest at the top of the distribution. It is not possible from this state of the world to construe an interpretation that allows 'thinning of the middle' *and* 'concentration' in the top and bottom deciles. It follows that in our hypothetical scenario the 'shape effect' histogram, Figure 2(b), and the resulting estimation of MRP+ of +0.288, could not constitute evidence of a 'hollowing out of the middle and a concentration in the highest and lowest deciles'. This would simply be a misreading; had the same 'shape effect' been produced by applying MMA to expenditure, or AMA to log-expenditure, then such a conclusion would be valid.

In summary, the 'shape effect' in Figure 2(b) and the resulting MRP+ of +0.288 are exactly what would be expected in our scenario, so as with Claim 1, the test of Claim 2 is whether the results are significantly different from Figure 2 (b) and MRP+=+0.288. It is not possible to draw valid conclusions, regarding the kind of distributional changes that we have discussed here, from the result of MRP+ without first calculating the expected value given changes in mean consumption and inequality. We now turn to the empirical literature and examine what these appropriate null hypotheses should be.

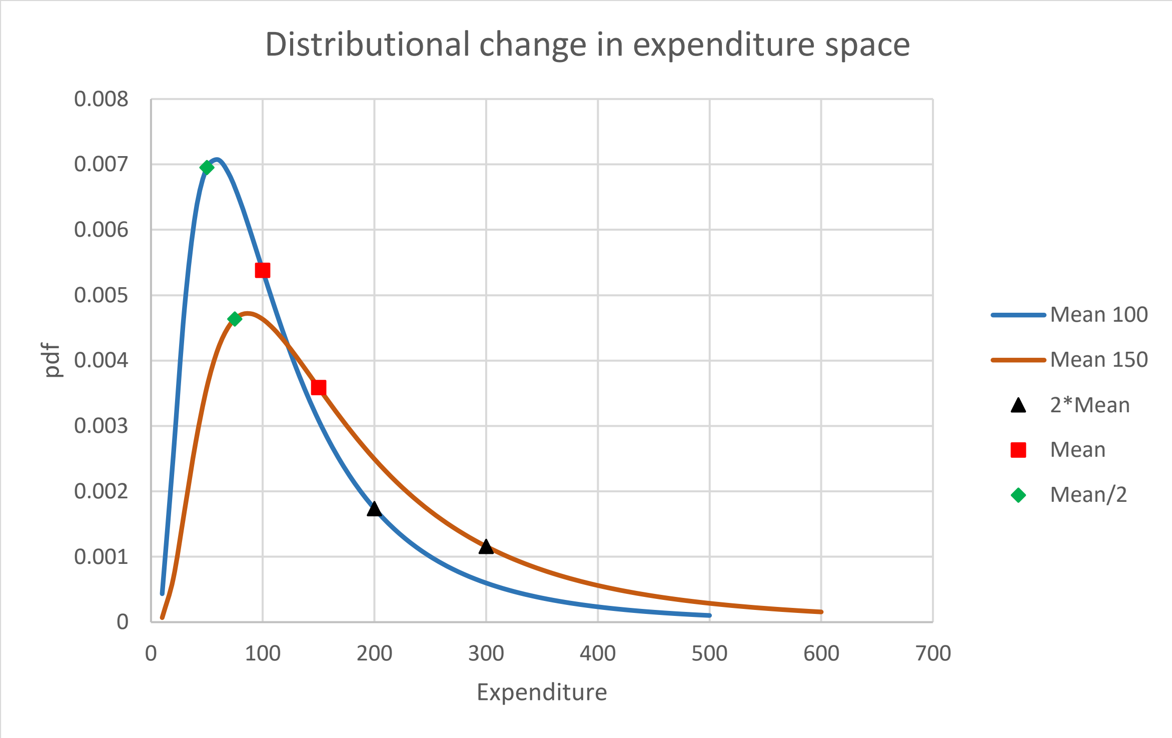

Figure 4*: The distribution change for 50% increase in mean consumption and unchanged Gini=0.40 in consumption space, as opposed to log-consumption space*

4. Empirical studies examined

We will now use the parametric approach, described in section 3, to generate 'shape effect' graphs of the type displayed in Figure 2 (b), and then calculate MRP+. As discussed, this process will yield the appropriate null hypotheses for testing Claim 1 and Claim 2. If the empirically estimated 'shape effects', which are the core of the CFM version of polarisation and resulting MRP+, are not significantly different from these null hypotheses then there is no basis for the conclusions drawn from them. It is important to note that we are not trying to replicate the CFM process; we do not have the CFM data. To the contrary, what we will show is that the CFM 'shape effect' graphs, and hence their estimated MRP+, do not depend on their data. The results are driven by the choice of AMA mechanism given a background of increasing mean expenditure. Before turning to CFM, however, we will first look at CMS, in which they examine the expenditure distribution in Ghana in more detail.

4.1 Clementi, Molini, and Schettino (2018)

CMS describe their findings as follows:

> "Looking at the results from 1991 to 2012, the paper documents how the distributional changes over time hollowed out the middle of the Ghanaian household consumption distribution and increased the concentration of households around the highest and lowest deciles; there was a clear surge in polarization indeed" (p.275).

We will now examine the evidence which forms the basis of this conclusion. CMS examine the Ghanaian household expenditure distribution, including data from 1991/2, 1998/9, 2005/6, and 2012/13. They report an overall increase in mean consumption between 1991/2 and 2012/13 of 92.1%, and an increase in the Gini index from 0.38 to 0.41 over the same period (CMS, Table 1, p.277). With these inputs, and assuming log Normality, the consumption distributions would be as illustrated in Figure 5. Following the procedure outlined above, we first produce the two median adjusted reference distributions, one using MMA and the other AMA, these decile 'cut points' are illustrated on Figure 5.[19] As in our theoretical example above, the large increase in mean consumption, combined with the election of AMA, causes the reference distribution to concertina inwards. In this case the first consumption level above the bottom decile would have to rise by 206% to avoid 'relegation', whereas the first consumption level below the top threshold would only have to rise by 34% to achieve 'promotion'. We then produce the relative density deciles, the 'shape effects', using MMA and AMA (Figures 6(a) and 6(b), respectively). Notice that Figure 6(b) is almost identical to the empirically derived 'shape effect' (CMS, Figure 1, p. 281); the decile data appear to fit with $R^2>0.99$.[20] Finally, we calculate the MRP and MRP+ for this theoretical transition from log Normal with mean consumption 459.91 and Gini=0.38, to log Normal with mean consumption 883.48 and Gini=0.41, the results are MRP=+0.061, and MRP+=+0.447.

From the 'shape effect' CMS conclude that the "U-shaped relative density is observed, indicating that polarization was hollowing out the middle of the Ghanaian household consumption" (p.281). But this 'shape effect' is exactly as would be expected given the change in mean and Gini, and assuming a log Normal distribution in 1991/2 and in 2012/13. As stated above, this does not constitute proof of log Normality, but it means that this

[19] Appendix B.2 outlines the derivation of these decile thresholds and tabulates them.
[20] The 'shape effect' histograms are compared alongside each other in Appendix D.1, panel (a).

relative density pattern, derived as it is under AMA, cannot constitute evidence that "sizable declines occurred in the middle" (CMS, p.282).

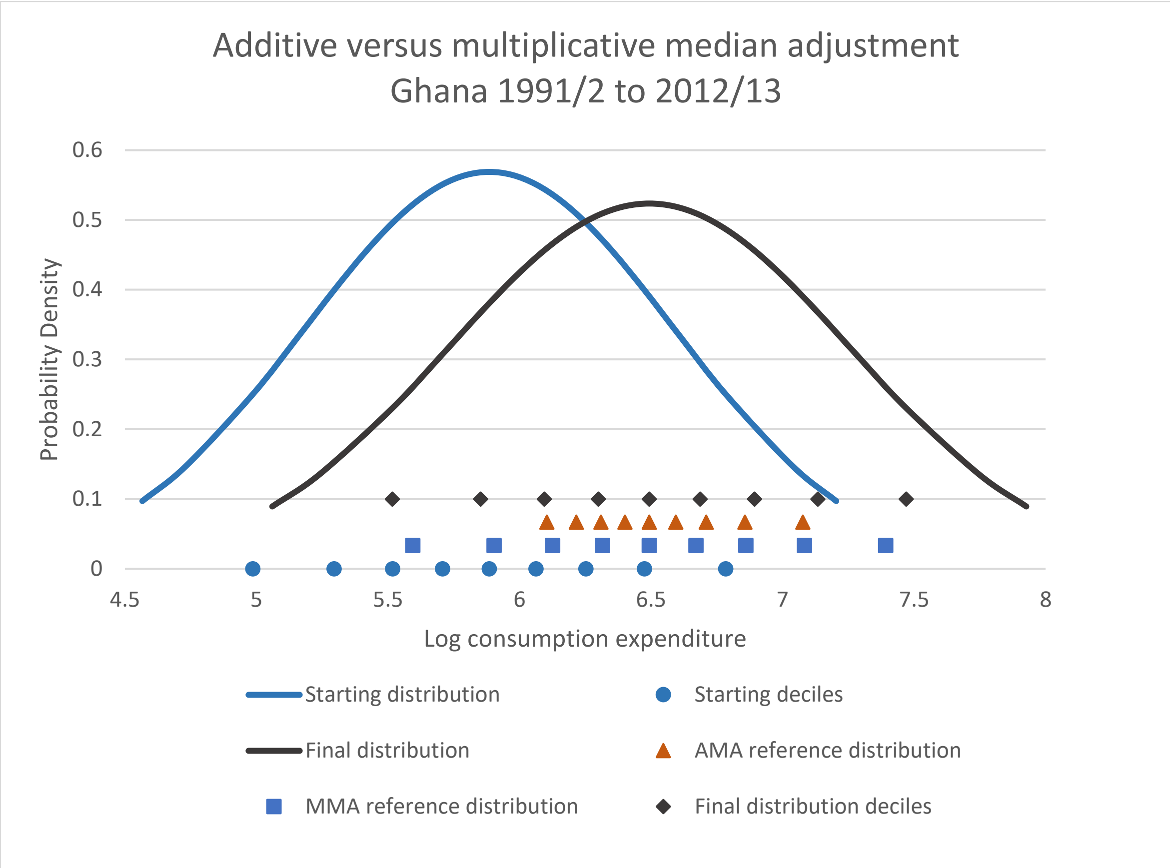

***Figure 5**: The effect on decile thresholds of the counterfactual reference distribution depending on whether AMA, or MMA, are employed.*

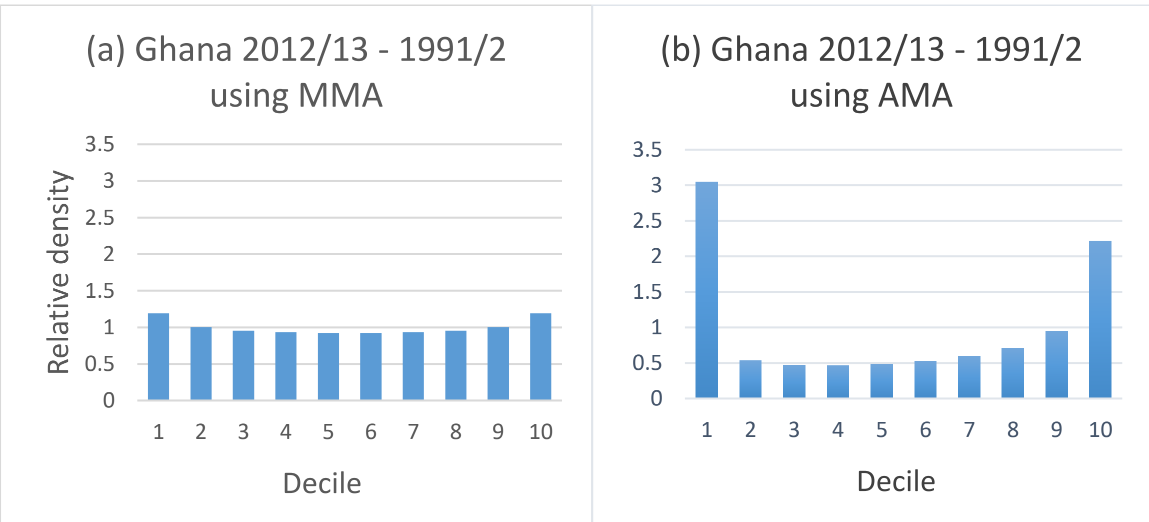

***Figure 6**: The theoretical decile realtive density histogram for Ghana (2012/13 – 1991/2) under the assumption of log Normality and employing (a) MMA, and (b) AMA.*

CMS do not report an overall MRP for the period from 1991/2 to 2012/13, only for the individual waves; however sequential MRP estimates are not additive, so we look to CFM where they report estimated MRP+=+0.44, insignificantly different from the +0.447 that is expected under the assumption of log Normality. CMS report a p-value of 0.00 for each of the three waves (CMS, Figure 2, p. 282) but this is relative to the "null hypothesis that the index equals 0". As discussed above, the correct null hypothesis for testing Claim 1 is MRP+=+0.447; MRP+=0 cannot be the appropriate null hypothesis, not least because it would be impossible. In case of an increase in mean consumption of 92.1% and an initial Gini of 0.38, it would be necessary for the Gini index to fall to 0.165 to achieve MRP+=0 (see Appendix C for the graphical display).[21] Not only do the 'shape effect' histogram and the resulting summary measure MRP+ not warrant the conclusions of 'hollowing out of the middle', but they also make these conclusions untenable.

4.2 Clementi, Fabiani, and Molini (2019)

There are two central claims in CFM of interest here: first, they claim that deploying the 'relative distribution' method allows them to identify "[d]istributional changes that went undetected by standard inequality measures" (p.408), and second, that these results allow them to account for the observed low growth elasticity of poverty in SSA. We will consider these claims in turn.

4.2.1 Distributional changes undetected by standard inequality measures

As we have seen, in section 4.1, the detailed study of the Ghanaian expenditure distribution does not offer any evidence of distributional changes that went undetected by standard inequality measures. The 'relative distribution' method returns exactly the 'shape effect' and resulting MRP+ that we would expect considering the change in Gini and, given that MRP+ is an absolute measure, the change in mean consumption. However, CFM cover 24 countries in SSA, so we now turn to these results to test the claims that "most countries

[21] Given a 92.1% increase in mean consumption, and an initial Gini of 0.38, it might be thought that an MRP+ of 0.00 would be achieved given a final Gini of 0.198 (i.e., 0.38/1.921), representing unchanged absolute inequality. However, since lower inequality means a rise in the ratio of the median to the mean, this would lead to larger median adjustments, the actual closing Gini would need to be 0.165 to generate an MRP+ of 0.00.

faced a significant process of downgrading" (p.417), "while the middle of the distribution hollows out" (p.418).

While CFM estimate the MRP+ for all 24 countries they select three representative countries for which they present the relative density histograms (p.419, Figure 4), one of which is Ghana, so we will begin by looking at their results for the other two, Ethiopia and South Africa. We repeat the process, outlined in section 3, and undertaken with respect to Ghana in section 4.1.

We display the results for Ethiopia, and South Africa in Figure 7(a) and 7(b). Inspection will show nearly identical results to the empirical non-parametric results displayed in CFM (p.419).[22] In fact, comparing the empirical population observed within each decile of the CFM reference distribution with estimates from our three-parameter model yields R-squared values of 0.99 for both Ethiopia and South Africa, just as it did in the case of Ghana. As in the case of Ghana, these 'shape effect' graphs (CFM, p.419), and the resulting MRP+ estimations cannot count as evidence of 'hollowing out' etc…, or of any 'changes in the distribution unaccounted for by standard inequality measures' since the results are exactly as expected given the change in mean expenditure and Gini. Based on these results CFM can obviously reject the null hypothesis of no change in MRP+, but they cannot reject the null hypothesis of 'no change in the distribution unaccounted for by the change in mean expenditure and Gini'.

For the remaining 21 countries, CFM tabulate the MRP+ (CFM, pp.432-3, Table A1) but do not display the relative density histograms; however, CFM report that results for a further 16 countries "closely replicate" these distributional changes. Notably, there are nineteen countries with significant growth in mean consumption over their respective sample periods. As CFM state, only in Madagascar and Zambia do they estimate significantly negative MRP+, notably the two countries with significant falls in mean consumption over the respective sample periods. We calculate the equivalent histogram for Madagascar. It is displayed in Figure 7(c), clearly showing the opposite characteristic of the other three, and again closely matching the data from CFM.[23]

[22] See Appendix D.1, panels (b) and (c), for a side-by-side comparison.
[23] The histogram for Madagascar was not included in the CFM published material but was kindly provided by the authors on October 21, 2019. See Appendix D.2 for a side-by-side comparison.

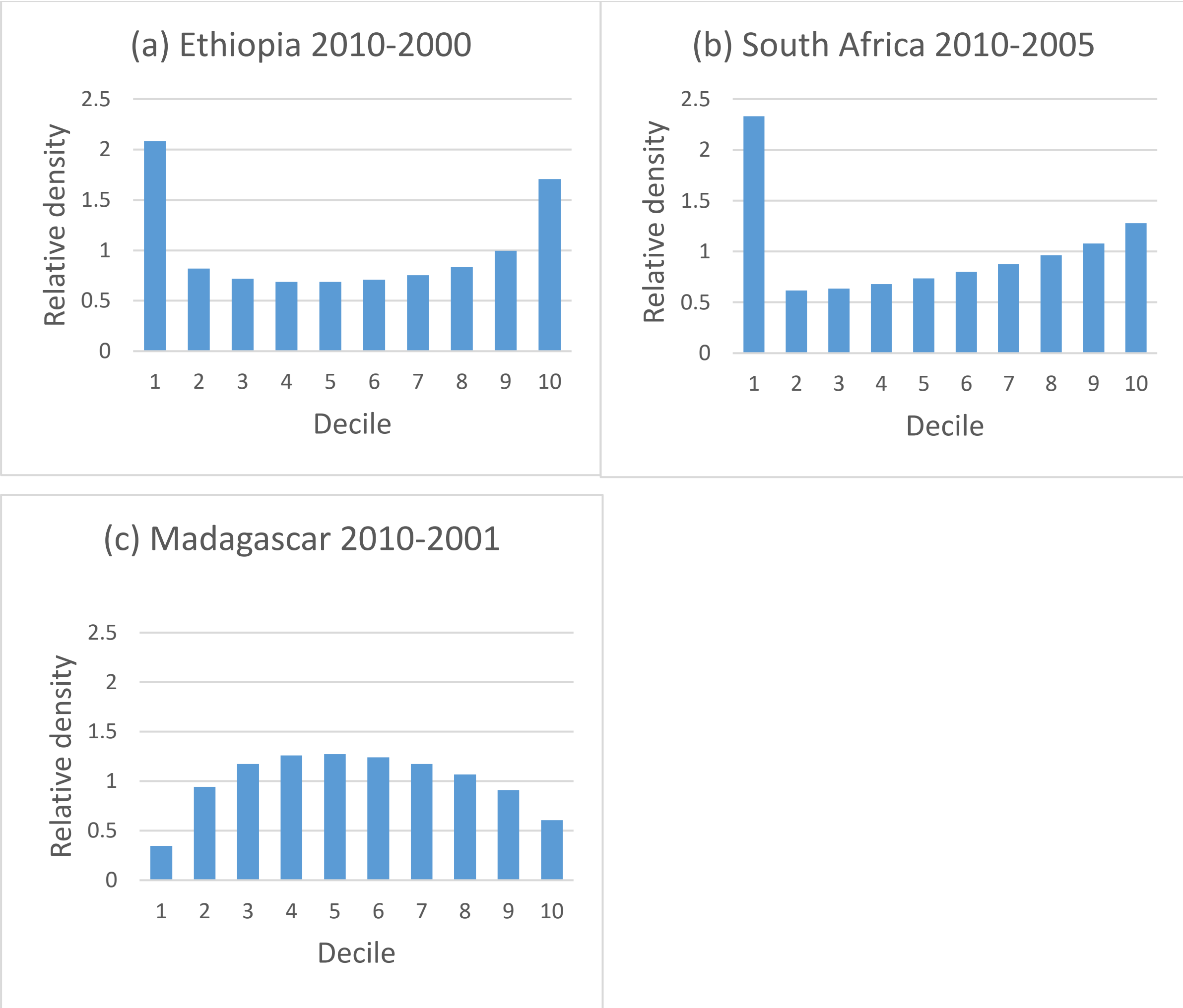

***Figure 7**: The theoretical decile relative density histograms, under the assumption of log Normality, for (a) Ethiopia (2010-2000), (b) South Africa (2010-2005), and (c) Madagascar (2010-2001).*

The empirical estimates for the 24 countries (CFM, pp.432-3, Table A1) are closely related to growth in mean expenditure. This is displayed in Figure 8; also displayed are the individual results of the parametrically obtained null hypotheses, given the growth and inequality data for each country.[24] Additionally, a curve showing the relationship between growth and the null hypothesis given unchanged Gini of 0.45 (the mid-range level of national inequality in the region) has been overlaid. The R-squared between the empirical estimates and the parametrically derived null hypothesis values is 0.92, and, as can be seen, the growth component is the dominant factor.[25] Certainly in the case of Côte d'Ivoire and Eswatini the

[24] The data are tabulated in Appendix E.
[25] It should be noted that calculating the appropriate null hypothesis for Zambia requires a manual adjustment to the decile cut between the first and second decile of the counterfactual reference distribution, to raise the expenditure level at the cut to zero. CFM will presumably have had to do the same with the expenditure distribution since applying the AMA, or in this case the subtractive median adjustment, results in 12% of the

estimated MRP+ is substantially higher than the calibrated null hypothesis, and there is evidence of 'downgrading' in these two countries. There is, however, no warrant for claims of widespread changes in the distributions unaccounted for by changes in mean consumption and inequality.

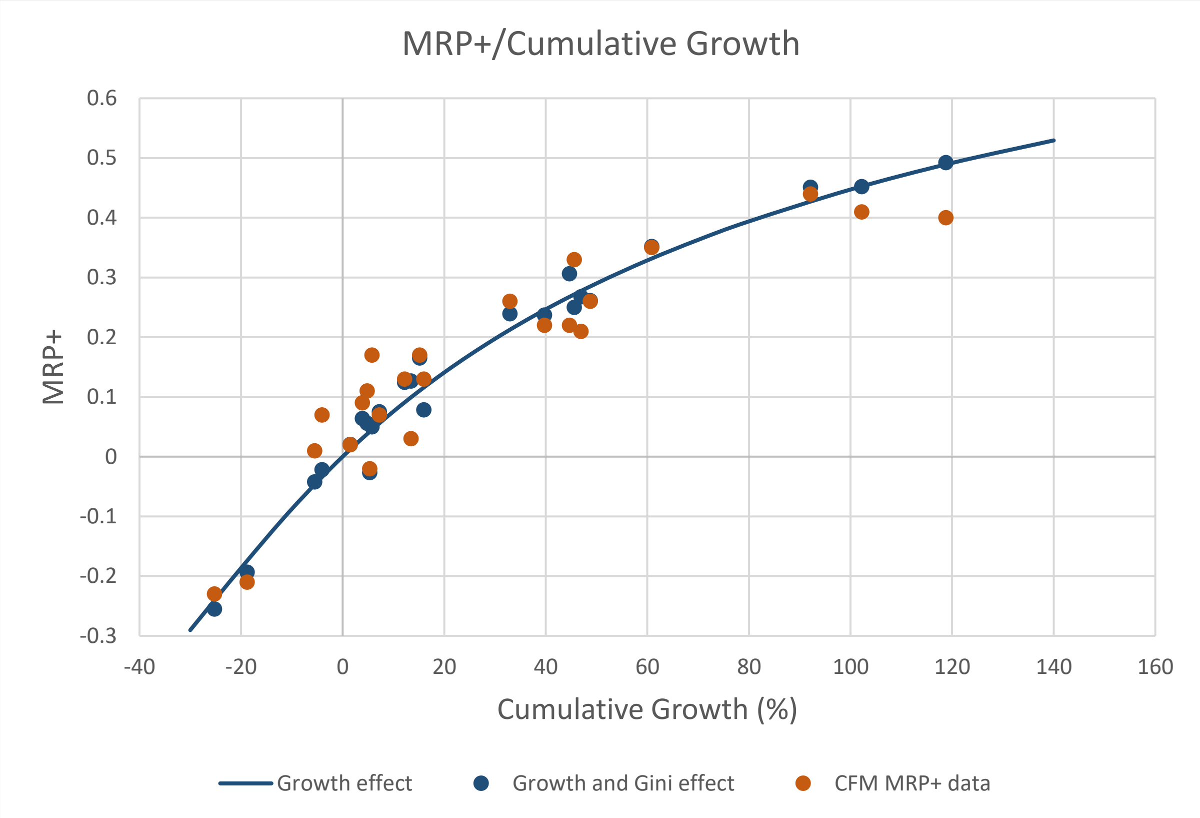

***Figure 8**: Cumulative growth/MRP+ for the 24 countries in the CFM sample. Growth data from CMS for Ghana, and from PovcalNet for the remaining 23, the MRP+ empirical results are from CFM. The 'Growth and Gini' effect data are generated using the procedure outlined in Section 3, and the results are tabulated, alongside the PovcalNet, CMS, and CFM data in Appendix D.*

CFM state, in their abstract, "[w]ithout this unfavourable redistribution, poverty could have decreased in these [19] countries by an additional five percentage points." But the implication of their result is rather that if the 'gains from growth' had been distributed evenly rather than proportionally there would have been an additional reduction in poverty of five percentage points amongst those 19 countries, with the obvious caveat about independence of growth and distribution. This observation is correct, but hardly new. And, as we would expect, in the case of falling earnings the opposite would be the case.

population having negative counterfactual expenditure. This demonstrates why support for translation-invariance rapidly evaporates in the event of falling means.

4.2.2 Mean consumption elasticity of poverty in the CFM 24

As we have just seen there is no evidence of widespread changes in the distributions of the 24 countries covered by CFM unaccounted for by changes in mean consumption and inequality. Now we will examine the actual evolution of poverty, relative to the changes in mean consumption and Gini, across the same sample.

As noted by Bourguignon (2003), and Epaulard (2003), for a log Normal distribution there is an 'identity' expressing the poverty headcount ratio (H) as a function of mean income and the standard deviation of log-income:

$$H = \Phi\left(\frac{\ln(Z/Y)}{\sigma} + \frac{\sigma}{2}\right) \tag{16}$$

where, Φ (.) is the cumulative standard Normal distribution, Z is the poverty line, Y is mean income, and σ is the standard deviation of log-income.[26] The standard deviation of log-income is in turn a function of the Gini index:

$$\sigma = \sqrt{2}\,\Phi^{-1}\left(\frac{Gini + 1}{2}\right) \tag{17}$$

where, Φ^{-1}(.) is the inverse of the cumulative standard Normal distribution. So, if the distribution is log Normal the poverty headcount ratio is a function of just two variables, mean income, and the Gini index.[27]

For the 24 countries in the CFM survey, we take the data for mean consumption, the Gini index, and the poverty headcount ratio from PovcalNet for the respective start dates and end dates. Comparison of the actual reported poverty headcount ratio with the theoretical level derived from equation (16) yields an R-squared of 0.987 for the start date cross-section, and 0.991 for the end date cross-section.[28] Using the individual country poverty headcount ratio, and the respective population data, from PovcalNet, indicates that the

[26] Recall from section 3, that if the entire shape effect matches that expected given log Normality then this constitutes evidence of log Normality. Noting that the entire shape effect, in case of the three countries for which the shape effect is provided in CFM, does match the effect expected under log Normality, with R-squared values of 0.99, and Appendix D.1 providing visual confirmation, we can assume that the consumption distributions are indeed close to log Normal.
[27] In keeping with Bourguignon (2003) we will refer to 'mean income' here when discussing the 'identity' model. However, when applying the 'identity' we will use mean consumption data. The 'identity' is still valid, in fact, consumption data is closer to log Normal than income data, as discussed above. The mechanics of the log Normal distribution are obviously unaffected.
[28] World Bank PovcalNet Database, 2018 Issue, retrieved in early 2020.The results are tabulated in Appendix F.

aggregate poverty headcount ratio for the sample fell from 58.8% for the start date cross section, to 46.9% for the closing cross section. Whereas, using the theoretical 'identity' model-based estimates indicates a fall from 58.0% to 46.5%. So, the evolution of poverty across this sample at least is exactly as expected given the changes in mean consumption and Gini index.

If the evolution of poverty is entirely accounted for by the change in mean consumption and inequality, perhaps it could be claimed, in line with the original motivation for Morris, Bernhardt, and Handcock (1994), that the relative distribution method is explanatory of increased inequality.[29] This would have to be 'standard', i.e., relative, inequality as measured by the Gini index since this is the explanatory variable in the poverty evolution account. However, this is problematic for two reasons. First, when initially introduced by Clementi and Schettino (2015), MRP+ was designed to explain distributional changes that had a deleterious effect on the poor *despite* inequality, as measured by the Gini index, falling. The second, more direct issue, is that for this sample there is no correlation between MRP+ and changes in Gini for the countries concerned. The relationship is displayed in Figure 9, the R-squared between MRP+ and percentage change in Gini is 0.002, and, in fact, the correlation is mildly negative -0.04.

[29] This is not what CFM claim, they claim that their methodology identifies precisely distributional changes that are not detected by 'standard' measures of inequality, not that it is explanatory of changes in 'standard' measures of inequality. However, for completeness we will consider this possibility here.

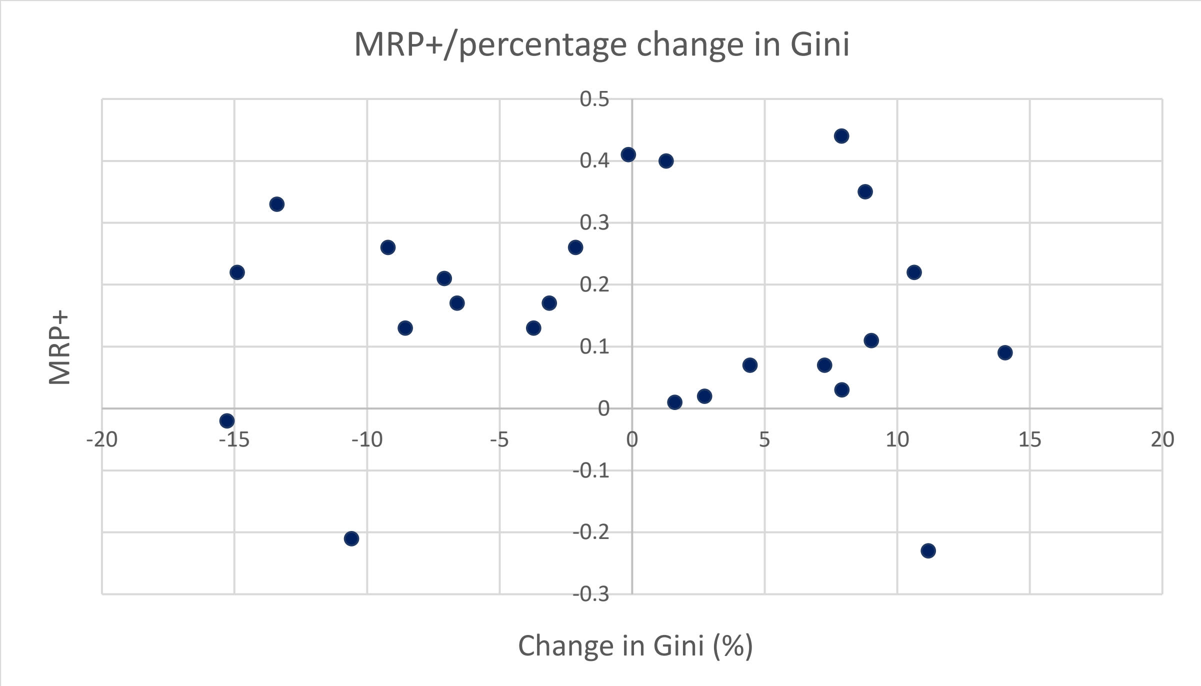

Figure 9*: Comparison of the empirically estimated MRP+, from CFM, with the percentage change in the Gini index, calculated using Gini data from PovcalNet for the 24 countries in the CFM sample.*

5 Conclusion

The 'relative distribution' methodology offers a useful tool; it is a distinct concept from the FW and ER concepts of polarisation, and it gives a level of insight into the details of distributional change that is not possible with a single summary measure. The issue here is with the detail of the implementation. As Handcock and Morris indicate it is up to the analyst to choose the location adjustment mechanism, but this must reflect the 'nature of the data'. When CFM elect to use AMA, they intentionally adopt a measure of absolute polarisation; once this is done the results must be interpreted accordingly.

Applying the MRP+ procedure to a hypothetical case in which we know *ex ante* that Claims 1 and 2 are false nonetheless leads to the rejection of the CFM null hypothesis of MRP+=0, so the procedure would indeed lead to the false conclusion that there had been changes in the distribution unaccounted for by changes in mean income and inequality, and that there had been a hollowing out of the middle combined with a concentration in the bottom and top deciles. We have shown that this is exactly what happens in CFM.

The issue is that outputs from a process that measures absolute polarisation are interpreted as if they were measuring relative polarisation. It is easy to understand how this misinterpretation can come about, which is exactly why this variation of the methodology should not be used. If the same shape effects identified by CFM had been the result of applying MMA to consumption data, or equivalently of applying AMA to log-consumption data, then that would have been evidence of hollowing out, and it would have offered evidence in the explanation of the consumption elasticity of poverty in SSA. However, as the results in CFM are precisely as expected, given the respective changes in mean consumption and inequality, they offer no contribution to the explanation of the evolution of poverty. The results do not diverge from the null hypotheses, once the null hypotheses are correctly calibrated. Not only does the evidence in CFM not support their conclusions, but it also precludes them.

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Appendix A The Q/Q-2 adjustment

The adjustment to the MRP when it is calculated from grouped data is simply a renormalisation so that the range remains the closed interval [-1, +1]. In the case of decile-based data for example, the maximum MRP is reached when the decile relative density vector is (5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5), in which case the MRP prior to the renormalisation adjustment would be +0.8. Adjusting by Q/Q-2, with Q=10 here, resets the maximum MRP to +1.

In the hypothetical scenario, described in section 3, the MRP for the continuous distribution is +0.2660. The MRP calculated from deciles is +0.2303 without the Q/Q-2 adjustment, and +0.2879 after the adjustment. For centiles, the equivalent results are 0.2635 and 0.2689 respectively, and for milliles, 0.2657 and 0.2662, respectively. It is not the case that the adjustment improves the estimation, although in case of this distribution the adjusted estimate derived from deciles is marginally closer to the continuous distribution result, it is an arbitrary renormalisation, just as in the factor of two included in the P^{W} relative to the P^{FW} metric. We simply need to ensure that the inclusion, or otherwise, is consistent with the methodology being compared.

Appendix B

B.1 Derivation of reference thresholds, decile relative densities, and MRP+ for the hypothetical scenario in Section 3

The hypothetical scenario in section 3 involves a rise in mean expenditure from 100 to 150, with constant inequality of Gini=0.40. So, the inputs for the calculations are:

$C_0 = 100$, and $G_0 = 0.40$, so $\sigma_0 = 0.7416$, and $M_0 = 75.96$

$C_t = 150$, and $G_t = 0.40$, so $\sigma_t = 0.7416$, and $M_t = 113.94$

and therefore $M_t - M_0 = 27.98$.

The initial decile thresholds are calculated according to equation (10), the counterfactual decile thresholds (under AMA per CFM) are then obtained by adding $M_t - M_0$ (which is ρ in CFM). These counterfactual thresholds are identified as the triangles in Figure 1.

The cumulative population with expenditure levels below the counterfactual thresholds is then calculated using equation (13). The resulting decile relative densities are the columns in the histogram Figure 2(b), from which the MRP+ of +0.288 is derived using equation (3).

***Table A1**: Counterfactual reference decile construction for an increase of 50% in mean consumption, and unchanged inequality, Gini=0.40.*

	Initial Decile Threshold	Counterfactual Decile Threshold	Cumulative Population	Decile Relative Density
1	29.36	67.34	23.914	2.391
2	40.69	78.67	30.874	0.696
3	51.48	89.46	37.219	0.634
4	62.95	100.93	43.506	0.629
5	75.96	113.94	50	0.649
6	91.66	129.64	56.910	0.691
7	112.06	150.04	64.476	0.757
8	141.79	179.77	73.069	0.859
9	196.49	234.46	83.475	1.041
10			100	1.653
			MRP+	+0.288

B.2 Derivation of reference thresholds, decile relative densities, and MRP+ for Ghana 1991/2 to 2012/13

The data for Ghana indicate an increase in mean consumption from 459.91 to 883.48 (data from CMS, Table 1, p.277), and an increase in Gini from 0.38 to 0.41 (ibid.). So, the inputs for the calculations are:

$C_0 = 459.91$, and $G_0 = 0.38$, so $\sigma_0 = 0.7012$, and $M_0 = 359.66$

$C_t = 883.48$, and $G_t = 0.41$, so $\sigma_t = 0.7620$, and $M_t = 660.85$

and therefore $M_t - M_0 = 301.19$.[30]

The initial decile thresholds are calculated according to equation (10), the counterfactual decile thresholds (under AMA per CFM) are then obtained by adding $M_t - M_0$ (which is ρ in CFM). These counterfactual thresholds are identified as the triangles in Figure 5. The cumulative population with expenditure levels below the counterfactual thresholds is then calculated using equation (13). The resulting decile relative densities are the columns in the histogram Figure 6(b), from which the MRP+ of +0.447 is derived using equation (3).

***Table A2**: Counterfactual reference decile construction Ghana 1991/2 to 2012/13.*

	Initial Decile Threshold	Counterfactual Decile Threshold	Cumulative Population	Decile Relative Density
1	146.42	447.61	30.458	3.046
2	199.33	500.52	35.769	0.531
3	249.00	550.18	40.497	0.473
4	301.12	602.31	45.156	0.466
5	359.66	660.85	50	0.484
6	429.59	730.77	55.250	0.525
7	519.51	820.70	61.190	0.594
8	648.94	950.13	68.312	0.712
9	883.45	1184.64	77.814	0.950
10			100	2.219
			MRP+	+0.447

[30] Notice that this AMA is derived from the theoretical values of the two medians. For comparison, the data given in CMS (Table 1, p.277) would indicate $\rho = M_t - M_0 = 302.94$.

Appendix C

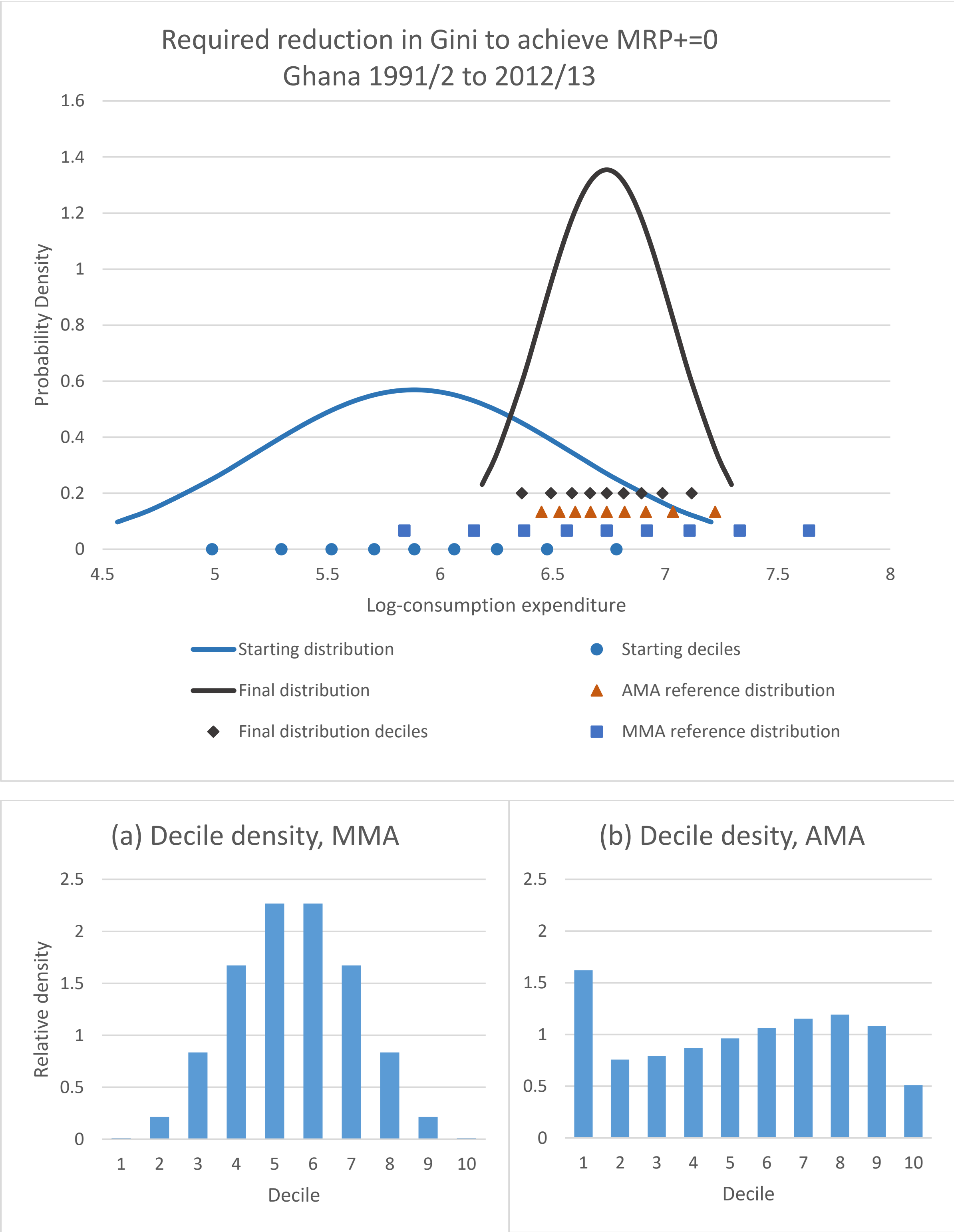

***Figure 10**: Required reduction in Gini to achieve MRP+=0, given 92.1% increase in mean consumption and an initial Gini of 0.38. Panel (a) equates to an MRP of -0.60, and panel (b) equates to an MRP+ of 0.00 as required.*

Appendix D

D.1 Relative density histograms for Ghana, Ethiopia, and South Africa

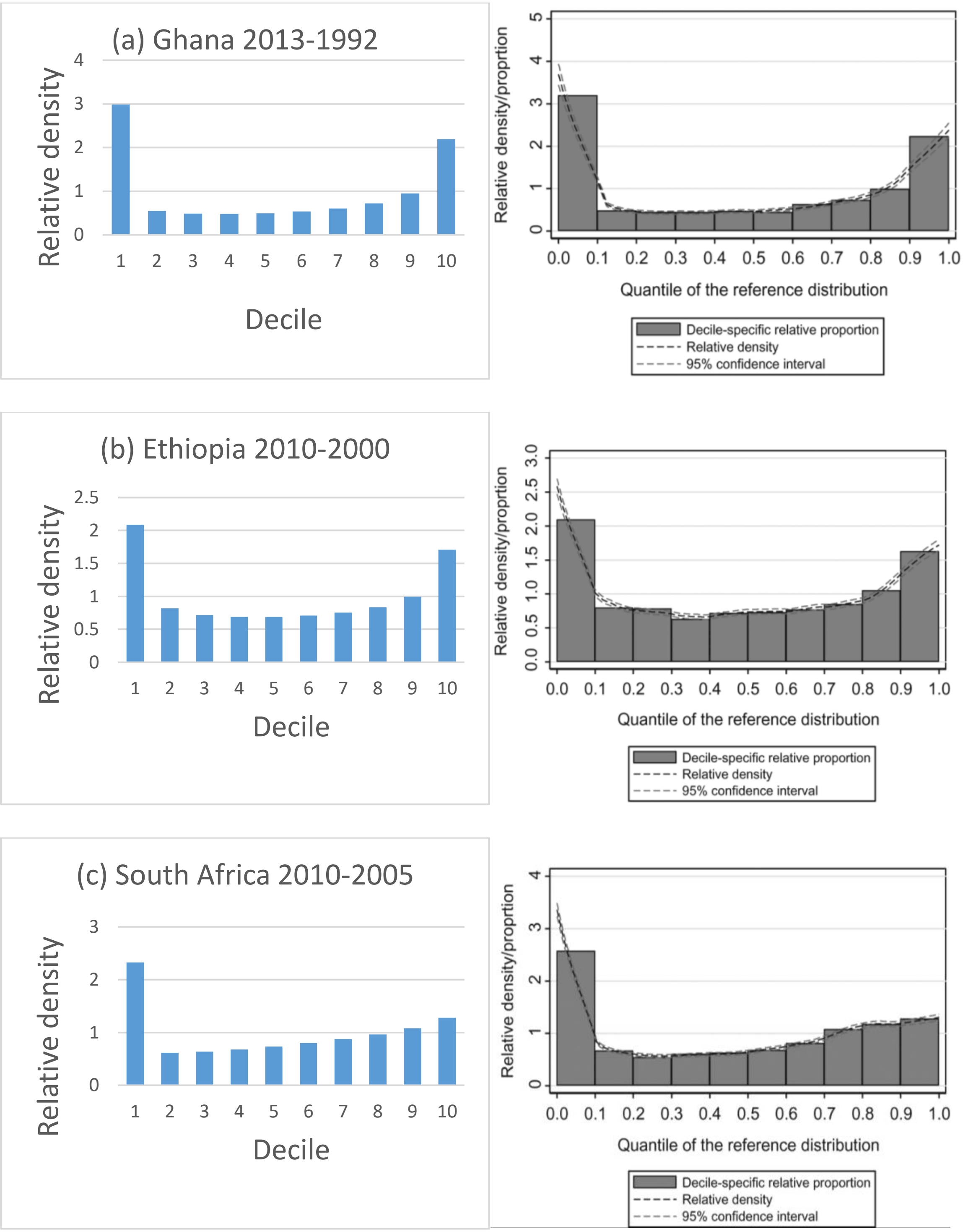

***Figure 11**: Our parametric estimate (left hand panels), and the empirical estimates (right hand panels) from CFM (p.419, Figure 4) for the relative distribution in (a) Ghana (2013-1992), (b) Ethiopia (2010-2000), and (c) South Africa (2010-2005).*

D.2 Relative density histograms for Madagascar

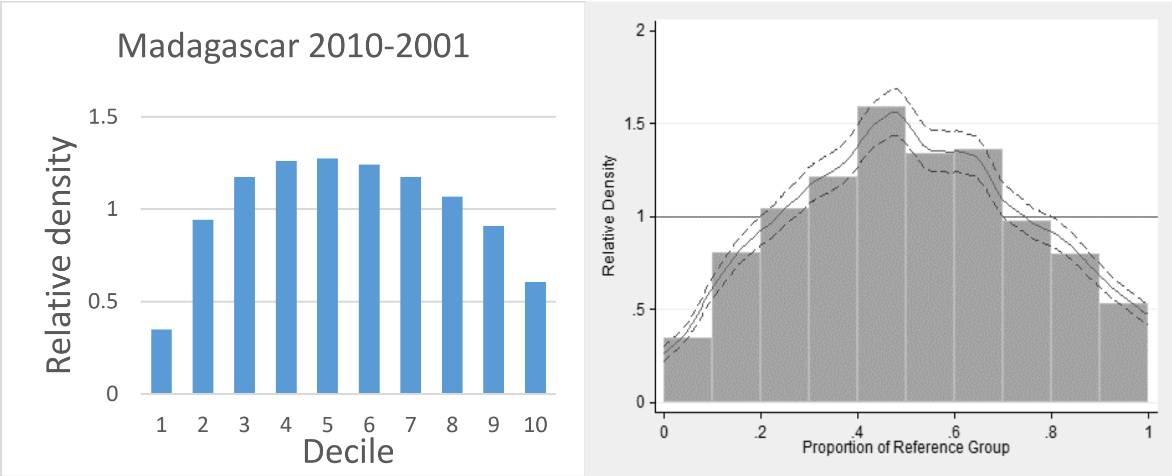

Figure 12*: Our parametric estimate (left hand panel), and the empirical estimates (right hand panel) from CFM for Madagascar (2010-2001). This histogram is not included in CFM's published article but was kindly provided by the authors on October 21, 2019.*

Appendix E

MRP+ null hypotheses compared with the CFM empirical estimates

***Table A3**: Comparison of the null hypothesis for MRP+, given changes in mean expenditure and inequality, and the empirical estimates from CFM. Mean consumption and Gini data from PovcalNet.*

	Growth in Mean Consumption (%)	Gini index		MRP+	
		Start	End	Parametric Null	Empirical
Botswana	15.2	0.6473	0.6046	0.1652	0.17
Burkina Faso	45.6	0.4994	0.4325	0.2502	0.33
Cameroon	-5.5	0.4214	0.4282	-0.0421	0.01
Chad	60.9	0.3982	0.4332	0.3519	0.35
Congo, DR	102.2	0.4216	0.421	0.4519	0.41
Côte d'Ivoire	-4.1	0.4134	0.4318	-0.0220	0.07
Eswatini	5.7	0.5311	0.5145	0.0500	0.17
Ethiopia	44.7	0.2998	0.3317	0.3061	0.22
Ghana	92.1	0.38	0.41	0.4466	0.44
Madagascar	-18.8	0.4744	0.4242	-0.1934	-0.21
Malawi	3.9	0.3987	0.4548	0.0639	0.09
Mauritania	16.0	0.3903	0.3569	0.0783	0.13
Mauritius	13.4	0.3565	0.3847	0.1265	0.03
Mozambique	39.7	0.5356	0.4558	0.2369	0.22
Namibia	12.2	0.6332	0.6097	0.1245	0.13
Nigeria	7.2	0.4006	0.4297	0.0751	0.07
Rwanda	46.9	0.4855	0.4511	0.2675	0.21
Senegal	1.5	0.3922	0.4029	0.0207	0.02
Sierra Leone	5.3	0.4017	0.3403	-0.0264	-0.02
South Africa	33.0	0.6476	0.6338	0.2395	0.26
Tanzania	118.8	0.373	0.3778	0.4924	0.40
Togo	4.8	0.4221	0.4602	0.0559	0.11
Uganda	48.7	0.4517	0.4101	0.2610	0.26
Zambia	-25.2	0.4913	0.5462	-0.2551	-0.23

RSQ= 0.9204

Appendix F

Poverty headcount ratio as a function of mean consumption and inequality

Table A4: Comparison of the theoretical poverty headcount ratio, as a function of mean consumption and inequality, and the observed ratio for the start year cross section of the CFM data set.

	Year	Mean Consumption[a]	Gini[a]	Poverty Headcount Theoretical[b]	Poverty Headcount Empirical[a]
Botswana	2002	252.13	0.6473	32.13	29.75
Burkina Faso	1998	49.31	0.4994	73.96	81.61
Cameroon	2001	131.46	0.4214	25.64	23.12
Chad	2003	61.01	0.3982	61.58	62.94
Congo, DR	2004	22.98	0.4216	94.13	94.05
Côte D'Ivoire	2002	123.8	0.4134	27.19	23.2
Eswatini	2000	110.87	0.5311	45.04	48.44
Ethiopia	1999	60.53	0.2998	57.38	61.25
Ghana	1991	76.04	0.3844	48.71	49.78
Madagascar	2001	59.05	0.4744	66.42	68.68
Malawi	2004	54	0.3987	67.74	73.41
Mauritania	2000	138.24	0.3903	19.82	19.59
Mauritius	2006	335.28	0.3565	0.91	0.42
Mozambique	1996	42.34	0.5356	79.31	82.85
Namibia	2003	211.34	0.6332	35.26	31.46
Nigeria	2003	70.76	0.4006	53.90	53.46
Rwanda	2000	53.72	0.4855	70.52	77.21
Senegal	2005	94.59	0.3922	37.56	37.44
Sierra Leone	2003	67.38	0.4017	56.57	60.58
South Africa	2005	269.65	0.6476	30.36	26.12
Tanzania	2000	35.93	0.373	84.95	85.96
Togo	2006	76.19	0.4221	51.65	55.55
Uganda	2002	67.92	0.4517	59.23	65.08
Zambia	1998	108.09	0.4913	41.94	42.14
				R SQ	0.988

Notes:

(a) Data from PovcalNet
(b) Calculated using equation (16)

__Table A5__: Comparison of the theoretical poverty headcount ratio, as a function of mean consumption and inequality, and the observed ratio for the end year cross section of the CFM data set.

	Year	Mean Consumption	Gini	Poverty Headcount Theoretical	Empirical
Botswana	2009	290.42	0.6046	22.88	18.24
B Faso	2003	71.8	0.4325	55.37	57.26
Cameroon	2007	124.21	0.4282	28.84	29.27
Chad	2011	98.14	0.4332	40.14	38.43
Congo, DR	2012	46.47	0.421	74.83	76.59
Cote D'Ivoire	2008	118.78	0.4318	31.21	29.14
Eswatini	2009	117.24	0.5145	41.11	42.03
Ethiopia	2010	87.57	0.3317	35.05	33.53
Ghana	2012	189.87	0.4237	13.33	12.05
Madagascar	2010	47.94	0.4242	73.60	78.47
Malawi	2010	56.11	0.4548	67.78	71.72
Mauritania	2008	160.32	0.3569	10.93	10.77
Mauritius	2012	380.37	0.3847	1.08	0.54
Mozambique	2008	59.17	0.4558	65.56	69.07
Namibia	2009	237.07	0.6097	28.95	22.60
Nigeria	2009	75.86	0.4297	52.45	53.47
Rwanda	2013	78.93	0.4511	52.20	56.84
Senegal	2011	95.98	0.4029	37.99	37.98
Sierra Leone	2011	70.98	0.3403	49.21	52.21
South Africa	2010	358.52	0.6338	21.48	16.53
Tanzania	2011	78.6	0.3778	46.26	49.09
Togo	2011	79.87	0.4602	52.37	54.18
Uganda	2012	101.02	0.4101	36.22	35.86
Zambia	2006	80.82	0.5462	58.41	60.46

R SQ	0.991