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Tracking error decision rules and accumulated wealth

NATHAN BERG[1] and DONALD LIEN*[2]

[1] *University of Texas at Dallas*
[2] *University of Texas at San Antonio*

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There is compelling evidence that typical decision-makers, including individual investors and even professional money managers, care about the difference between their portfolio returns and a reference point, or benchmark return. In the context of financial markets, likely benchmarks against which investors compare their own returns include easy-to-focus-on numbers such as one's own past payoffs, historical average payoffs, and the payoffs of competitors. Referring to the gap between one's current portfolio return and the benchmark return as 'tracking error', this paper develops a simple model to study the consequences and possible origins of investors who use expected tracking error to guide their portfolio decisions, referred to as 'tracking error types'. In particular, this paper analyses the level of risk-taking and accumulated wealth of tracking error types using standard mean-variance investors as a comparison group. The behaviour of these two types are studied first in isolation, and then in an equilibrium model. Simple analytic results together with statistics summarizing simulated wealth accumulations point to the conclusion that tracking error—whether it is interpreted as reflecting inertia, habituation, or a propensity to make social comparisons in evaluating one's own performance—leads to greater risk-taking and greater shares of accumulated wealth. This result holds even though the two types are calibrated to be identically risk-averse when expected tracking error equals zero. In the equilibrium model, increased aggregate levels of risk-taking reduce the returns on risk. Therefore, the net social effect of tracking-error-induced risk-taking is potentially ambiguous. This paper shows, however, that tracking error promotes a pattern of specialization that helps the economy move towards the path of maximum accumulated wealth.

1. Introduction

Tracking error is the difference between a portfolio's return and some benchmark level of performance. Decision-makers who care about tracking error compare their own performance to a benchmark such as a market-index, a historical level of aggregate performance, or their own recent portfolio return. Thus, relative performance emerges as a key criterion by which investors make judgments about the value of a portfolio. Of course, being concerned about one's relative performance does not completely

Berg: Cecil and Ida Green Assistant Professor of Economics, University of Texas at Dallas, Richardson, Texas. Phone 1-972-883-2088, Fax 1-972-883-2735, Email nberg@utdallas.edu

Lien: Professor of Economic, University of Texas at San Antonio, San Antonio, Texas. Phone 1-210-458-7312, Fax 1-210-458-5837, Email dlien@utsa.edu

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displace the risk and return considerations which play a central role in traditional analyses of portfolio choice. Therefore, this paper allows the weight placed on 'the pursuit of high relative performance' to vary in magnitude. The importance of tracking error relative to the desire for high expected return and low variance may be low, high, or somewhere in the middle. In fact, one of the central objectives of this paper is to analyse the effect, in terms of risk-taking and accumulated wealth, of tracking error (i.e. a preference for *relative* performance) as its subjective importance ranges from low-to high-priority status.

There is considerable evidence that many financial market professionals (Locke and Mann, 1999; Coval and Shumway, 2001) and perhaps most individual investors (Heisler, 1996; Odean, 1998) are concerned about tracking error when making portfolio decisions. In mutual fund advertisements and in daily reporting of market activity in the *Wall Street Journal*, comparisons between funds and market indices are commonplace. In such reports, money managers sometimes comment that their losses are made less painful due to the fact that they are ahead of their fund category's index. This suggests that, in the minds of some fund managers, losing $100 on a day in which everyone else loses $200 is quite different from losing $100 on a day when everyone else breaks even.

In addition to considerable anecdotal evidence pointing to the importance of tracking error considerations in modern financial markets, experimental evidence likewise has called attention to the fact that typical human preferences are more sensitive to departures from reference points than to absolute levels of consumption (Rabin, 1996; Rabin and Thaler, 2001). The status quo effect, 1
referring to reversing one's ranking of alternatives depending on what one currently owns, appears to be a robust component of typical preference orderings (Thaler and Johnson, 1990). Economists drawing on the evolutionary biology literature have also noted that even our physical senses are more keen to detect departures from the status quo than to accurately identify levels, e.g. in temperature, duration, pressure, volume, or odour, (see discussions and references in Gintis, 2000). This gives a tentative biological basis to the reference-point theory of choice.

The academic finance literature has studied the problem of how to minimize a (symmetric) tracking error objective, starting with Roll (1992), where controlling 'tracking error' was analysed as the explicit objective of a money manager trying to match the performance of a benchmark such as the S&P 500. Rudolf *et al*. (1999) investigated asymmetric extensions of the tracking error minimization problem, considering lower partial moment objectives and min-max objectives with one-sided deviations. The professional finance world has also weighed in on aspects of portfolio management when the objective is to control tracking error, as in the studies of Lee (1998), Gupta *et al* (1999), and Baierl and Chen (2000).

Rather than focusing on the problem of how to efficiently control tracking error, this paper attempts to analyse the long-run aggregate consequences of tracking error decision-making itself. In particular, this paper addresses the question of how tracking error decision-makers perform relative to mean-variance decision-makers who do not care about relative performance. The long-run levels of accumulated wealth—of tracking error types and mean-variance types—are compared, first in isolation, and then in an equilibrium environment where the decisions of tracking error types affect the choices of mean-variance types through the price mechanism. In addition to analysing the relative performance of tracking error types in terms of their share of accumulated wealth, the effect of tracking error types on the accumulation of economy-wide aggregate wealth is also considered.

Before presenting the model and deriving the main results, it is worthwhile to consider the connections between the notion of 'tracking error' contemplated in this paper and the growing literature on 'loss aversion'. To avoid confusion over the multiple meanings which are frequently associated with the term, Lien (2001) distinguishes between 'strong loss aversion', which refers to individuals who are more sensitive to losses than to gains (with respect to a reference-point level of wealth), and 'weak loss aversion', which refers to risk-loving attitudes over losses together with risk aversion over gains. Both components of loss aversion are present in Kahneman and Tversky's (1979) pioneering paper on 'prospect theory' of which loss aversion was a key ingredient.

The main connection between tracking error and loss aversion is the prominent role played by the reference point that comes into play when individuals subjectively evaluate risk. Both theories draw on a combination of experimental psychology and econometric analysis of financial market data to hypothesize that individuals are sensitive to deviations from particular reference points—often the status quo—and not merely to deviations from the mean. There are important differences, however, which are, perhaps, easiest to describe in terms of the shape of investors' utility-of-wealth function.

Strong and weak loss aversion have clear-cut implications about the shape of the utility function. The asymmetric sensitivity to gains and losses identified with strong loss aversion corresponds to a 'kinked' utility function, where left- and right-sided derivatives are unequal at the reference point. And the risk-loving/risk-averting combination referred to as weak loss aversion, of course, translates into a utility function that is convex to the left of the reference point and concave to the right.

In contrast, tracking error preferences (as they are specified in this paper, at least) do not share either of those features in general. The utility function specifications in this paper are smooth at the reference point and are not necessarily concave or convex anywhere on their domains. These differences in the shape of the utility function highlight the theoretical distinction between tracking error and loss aversion. The tracking error approach focuses solely on the reference point aspect of preferences which it shares in common with loss aversion, without committing to additional hypotheses about discontinuities or convexity. Despite the theoretical differences, then, this paper should be seen as complementary to loss aversion models, in seeking to study the causes and consequences of reference point (tracking error) preferences in the financial market context.

2. Comparing the wealth accumulations of tracking error types and mean-variance types in isolation

Each individual must choose the share, x_t, of current wealth W_t to be allocated to a risky activity with gross return R_{t+1}. The remaining share, $1-x_t$, is then allocated to a safe activity with risk-free gross return S. Wealth evolves according to

$$W_{t+1} = W_t[x_t R_{t+1} + (1-x_t)S]$$

Define gross total return as

$$Z_{t+1} \equiv \frac{W_{t+1}}{W_t} = x_t R_{t+1} + (1-x_t)S$$

Assume the risky return process $\{R_t\}$ is distributed according to

$$\log(R_{t+1}) = \rho \log(R_t) + \varepsilon_{t+1},\ R_0 \equiv e^{\mu+\frac{\sigma^2}{2}}$$

where the sequence $\{\varepsilon_{t+1}\}$ is i.i.d. normal $N(\mu, \sigma^2)$, and $\rho \in [0, 1]$ is a parameter indicating time persistence in the returns process. The returns process, itself, $(\{R_t\})$ can be related to an underlying asset price process $\{P_t\}$:

$$R_{t+1} = \frac{P_{t+1}}{P_t}$$

For $\rho = 0$, the return process is uncorrelated Gaussian noise and the price process is (discrete time) geometric Brownian motion. For $\rho = 1$, $\{R_t\}$ itself is (discrete time) geometric Brownian motion. Intermediate values of $\rho \in (0, 1)$ allow for various degrees of persistence in the returns process, i.e. successful generations of investors tend to be followed by successful generations, and so forth.

These discrete time wealth and returns processes are conceived of as discrete approximations corresponding to continuous time processes. Next, those corresponding continuous time processes are analysed directly in order to motivate the subsequent specification of preferences in terms of expected utility functions. The key point is that the preferences of both TE and MV types depend on the mean and variance of total gross returns rather than on moments of the absolute level of wealth. In order to derive this as an expected utility function, the wealth process is written in continuous time before being translated back into discrete time. The discrete time setting is more intuitive and is a practical necessity for implementing the simulation analysis presented later. However, the continuous time setting better facilitates the computation of closed-form moments which, in turn, simplify the derivation of an expected utility objective.

Translating to continuous time now, the returns process is specified as simple Brownian motion with time trend μ:

$$\mathrm{d}R = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z$$

where z is simple Brownian motion with $E\,\mathrm{d}z = 0$ and $\operatorname{var}\mathrm{d}z = \mathrm{d}t$. Writing the net safe return as $s \equiv S - 1$, wealth evolves according to:

$$\mathrm{d}W = W(t)[x(t)\mathrm{d}R + (1 - x(t))s\,\mathrm{d}t] = W(t)[(x(t)(\mu - s) + s)\mathrm{d}t + x(t)\sigma\,\mathrm{d}z]$$

Thus, $W(t)$ is seen to be geometric Brownian motion. This implies that $\log[W(t)]$ is simple Brownian motion:

$$d\log[W(t)] = \left[x(t)(\mu - s) + s - \frac{x(t)^2\sigma^2}{2}\right]\mathrm{d}t + x(t)\sigma\,\mathrm{d}z$$

Assuming $x(\cdot)$ remains constant on $[t, t+\tau]$, the following finite increment is a normal random variable:

$$\log[W(t+\tau)] - \log[W(t)] \sim N\left\{\left[x(t)(\mu - s) + s - \frac{x(t)^2\sigma^2}{2}\right]\tau,\ x(t)^2\sigma^2\tau\right\}$$

Conditional on $W(t)$, $W(t+\tau)$ is log-normal:

$$\log[W(t+\tau)] \mid W(t) \sim \mathrm{N}\left\{\left[x(t)(\mu - s) + s - \frac{x(t)^2\sigma^2}{2}\right]\tau + \log[W(t)], x(t)^2\sigma^2\tau\right\}$$

Finally, assuming that risk preferences are represented by the constant relative risk aversion expected utility function

$$u(W)=\frac{W^{1-\beta}}{1-\beta}$$

with risk aversion parameter $\beta>0$ and $\beta\neq 1$, it follows that, conditioning on time t information, $u(W(t+\tau))$ is also log-normal:

$$\log[u(W(t+\tau))]|W(t)=\{(1-\beta)\log[W(t+\tau)]-\log(1-\beta)\}|W(t)\sim$$
$$\mathrm{N}\left[(1-\beta)\left\{\left[x(t)(\mu-s)+s-\frac{x(t)^2\sigma^2}{2}\right]\tau+\log[W(t)]\right\}-\log(1-\beta),\ (1-\beta)^2x(t)^2\sigma^2\tau\right]$$

Taking expectations, the following mean-variance objective function finally emerges:

$$E_tu(W(t+\tau))=\frac{W(t)^{1-\beta}}{1-\beta}e^{(1-\beta)\tau\left[x(t)(\mu-s)+s-\frac{\beta}{2}x(t)^2\sigma^2\right]}$$

Maximizing $E_tu(W(t+\tau))$ with respect to $x(t)$ is seen to be equivalent to maximizing

$$x(t)(\mu-s)+s-\frac{\beta}{2}x(t)^2\sigma^2\approx\left[E_t(Z_{t+\tau})-1-\frac{\beta}{2}\mathrm{var}_t(Z_{t+\tau})\right]\frac{1}{\tau} \qquad (1)$$

where $Z_{t+\tau}\equiv\frac{W(t+\tau)}{W(t)}$ is the gross return on the time t decision after the elapsed duration $\tau>0$. To see this, use the approximation

$$Z_{t+\tau}=\frac{W(t+\tau)-W(t)}{W(t)}+1\approx\frac{\mathrm{d}W(t)}{W(t)}+1=[x(t)(\mu-s)+s]\mathrm{d}t+x(t)\sigma\,\mathrm{d}z+1$$

Setting $\mathrm{d}t=\tau$, the first two moments are

$$E_tZ_{t+\tau}=[x(t)(\mu-s)+s]\tau,\ \text{and}\ \mathrm{var}_tZ_{t+\tau}=x(t)^2\sigma^2\tau$$

This demonstrates the rationale behind the approximation in (1). Relying on the approximation in (1), this paper proceeds in discrete time, specifying mean-variance preferences in terms of the mean and variance of *total gross return* Z rather than in the moments of the absolute level of wealth W.

2.1 *The mean-variance (MV) type's utility function*

MV types seek high expected return and low variance, reflected in the standard mean-variance objective function

$$u_t^{\mathrm{MV}}(Z_{t+1})=E_t(Z_t+1)-\frac{\beta}{2}\mathrm{var}_t(Z_t+1) \qquad (2)$$

where the t subscript on u_t^{MV} reflects the fact that the utility function operates on its argument with respect to period t information. The MV type's utility function was derived above in continuous time within the expected utility framework under the assumption of constant relative risk aversion. This set-up allows portfolios to be evaluated in terms of percentage return (rather than level of wealth), avoiding the difficulty of dealing with risk attitudes that change with the level of wealth.

From the derivation above, the time t distribution of log (R_{t+1}) is normally distributed, i.e.

$$\log(R_{t+1})|\text{period } t \text{ information} \sim \mathrm{N}\left(\rho\log(R_t)+\mu,\, \sigma^2\right)$$

The MV type's objective function may be developed as follows:

$$\begin{aligned} u_t^{\mathrm{MV}}(Z_{t+1}) &= E_t(Z_{t+1}) - \frac{\beta}{2}\mathrm{var}_t(Z_{t+1}) \\ &= E_t[x_t R_{t+1} + (1-x_t)S] - \frac{\beta}{2}\mathrm{var}_t[x_t R_{t+1} + (1-x_t)S] \\ &= S + \left(R_t^\rho e^{\mu+\frac{\sigma^2}{2}} - S\right)x_t - \frac{\beta}{2}R_t^{2\rho}e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)x_t^2 \end{aligned}$$

Choosing $x_t \in [x, \bar{x}]$ to maximize (2), the optimal choice, x_t^{MV}, is the function

$$x_t^{\mathrm{MV}} = \begin{cases} x^* & \text{if } x^* \in [\underline{x}, \bar{x}] \\ \bar{x} & \text{if } x^* > \bar{x} \\ \underline{x} & \text{if } x^* < \underline{x} \end{cases}$$

where

$$x^* \equiv \frac{R_t^\rho e^{\mu+\frac{\sigma^2}{2}} - S}{\beta R_t^{2\rho} e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)}$$

Different values of $[\underline{x}, \bar{x}]$, ranging from $[-\infty, \infty]$ to [0, 1], correspond to different assumptions about the institutional constraints on short-selling R, and on borrowing S (taking a leveraged long position in R).

The MV type's optimal choice has the same sign as the difference between the expected risky return and safe return. In line with intuition, the MV type wants to fully exploit the gap in expected returns by taking an infinite position as return risk approaches zero, i.e. $\left|x_t^{\mathrm{MV}}\right| \to \infty$ as $\sigma^2 \to 0$. Less intuitive, however, is the effect of an increased degree of persistence in the returns process, reflected in the parameter ρ. It can be shown that

$$\mathrm{sign}\left[\frac{\mathrm{d}x_t^{\mathrm{MV}}}{\mathrm{d}\rho}\right] = \mathrm{sign}\left[2S - R_t^\rho e^{\mu+\frac{\sigma^2}{2}}\right]$$

Thus, the effect of more persistence on x_t^{MV} is positive unless the expected risky return $e^{\mu+\frac{\sigma^2}{2}}$ is significantly higher than the safe return S. Increased persistence raises both expected return and variance. When the expected risky return is relatively close to the safe return, the mean effect dominates, leading to a higher quantity demanded of R. But when the expected risky return is already more than twice as big as the safe return, the variance effect dominates, damping the demand for risk.

2.2 *Tracking error (TE) type's utility function*

Like MV types, TE types seek high expected return and low variance. But in addition to mean and variance, TE types also care about beating the benchmark return, denoted Z_{bt}. Three alternative benchmarks are specified subsequently, each corresponding to a psychological phenomenon which has been suggested elsewhere in the finance and economics literature. Social comparisons (Abel, 1990) may lead investors to focus on the returns of competitors, as in the saying about 'keeping up with the Joneses'. Alternatively, habituation to past levels of return may cause investors to evaluate today's portfolio returns in relation to returns from the recent past (Constantinides, 1990; Shi and Epstein, 1993). And a number of other psychological phenomena cited in Rabin and Thaler (2001), most notably, loss aversion, may lead investors to focus on other reference points.

TE types' preference for tracking error is represented by the following utility function:

$$u_t^{\mathrm{TE}}(Z_{t+1}) = E_t(Z_{t+1}) - \frac{\beta^{\mathrm{TE}}}{2}\mathrm{var}_t(Z_{t+1}) + \gamma\left[E_t(Z_{t+1} - Z_{\mathrm{b}t})^{\lambda} - \frac{\phi}{2}\mathrm{var}_t(Z_{t+1} - Z_{\mathrm{b}t})\right] \tag{3}$$

where $\lambda \in \{1, 3, 5, \ldots\}$ is a positive, odd integer (reflecting asymmetric subjective evaluations of gains versus losses relative to Z_{bt}); $\gamma > 0$ measures the strength of tracking error considerations relative to mean-seeking and risk-averting motives; and ϕ measures the degree to which the TE type seeks to avoid tracking error volatility.[1]

The TE utility function (3) contains two special cases which we now consider in greater detail. Without imposing any restrictions on (3), the TE types' degree of risk aversion β^{TE} may be larger or smaller than the MV types' risk aversion parameter β. In order to cleanly examine the effect of the tracking error term on risk-taking, however, it will be appropriate to first normalize the risk aversion parameters somehow so that the two types begin on a 'level playing field' regarding their propensities to take risk. We argue that, instead of the restriction $\beta^{\mathrm{TE}} = \beta$, the alternative restriction

$$\beta^{\mathrm{TE}} + \gamma\phi = \beta \tag{4}$$

makes more sense. This claim is justified as follows.

Inspecting the TE objective (3), one may note that, because Z_{bt} is non-stochastic at time t and therefore, because $\mathrm{var}_t(Z_{t+1} - Z_{\mathrm{bt}}) = \mathrm{var}(Z_{t+1})$, the coefficient on portfolio risk is actually

$$\frac{1}{2}(\beta^{\mathrm{TE}} + \gamma\phi) \tag{5}$$

Naively equating β and β^{TE} leads to a pair of utility functions in which TE types are always more risk averse, even when $E_t(Z_{t+1} - Z_{\mathrm{b}t})^{\lambda} = 0$ and the tracking error term exerts no effect.

TE types avert risk for two reasons. First, TE types are risk averse like MV types are and intrinsically dislike $\mathrm{var}_t(Z_{t+1})$. But return risk is undesirable for a second reason as well: it makes one's own performance relative to the benchmark less certain. MV types, on the other hand, experience only the first of these motives. The restriction (4) sets the magnitude of TE types' risk aversion (due to each of the two motives) equal to the MV types' risk aversion parameter. Restriction (4) implies that MV and TE types are equally concerned about the volatility of their returns, albeit for different reasons.

The goal in considering different parameterizations of risk aversion is to construct an intuitive 'level playing field' so that the effect of tracking error preferences on risk-taking can be analysed in

[1]All of the results reported in this paper are valid under an alternative specification of the utility function where the term $[E_t(Z_{t+1} - Z_{\mathrm{b}t})^{\lambda} - \phi\mathrm{var}_t(Z_{t+1} - Z_{\mathrm{b}t})]$ is replaced with $\left[E_t\left(\frac{Z_{t+1}}{Z_{\mathrm{b}t}} - 1\right)^{\lambda} - \phi\mathrm{var}_t\left(\frac{Z_{t+1}}{Z_{\mathrm{b}t}} - 1\right)\right]$. Both formulations capture the essence of TE-type behaviour: gambles are preferred when they are expected to exceed the benchmark ($E_t\frac{Z_{t+1}}{Z_{\mathrm{b}t}} > 1$ or $E_t Z_{t+1} > Z_{\mathrm{bt}}$), and less preferred otherwise.

isolation from the confounding effects of *a priori* differential degrees of risk aversion. In fact, a decomposition of the tracking error term $E_t(Z_{t+1}-Z_{bt})^{\lambda}$ (provided shortly) demonstrates that its effect on risk-taking is, for $\lambda \geq 3$, ambiguous. Whether or not the tracking error term leads to more or less risk-taking over time is one of the central questions in this paper. Therefore, we argue that (4) better calibrates the risk attitudes of the two types so that neither one begins with a greater propensity to take risk.

In fact, most of the numerical results which follow were qualitatively re-produced under the alternative assumption that $\beta^{TE}=\beta$, although with less noticeable differences and more auxiliary parameter restrictions required. Of course, the restriction $\beta^{TE}=\beta$ disadvantages TE types in terms of accumulating wealth, since it burdens them with a greater overall degree of risk aversion relative to MV types: $\beta^{TE}=\beta \Rightarrow \beta^{TE}+\gamma\phi > \beta$. Therefore we argue that the restriction (4) offers a cleaner comparison. The restriction (6) is assumed to hold throughout the remainder of the paper, yielding the following simplified expression for TE types' utility:

$$u_t^{TE}(Z_{t+1})=u_t^{MV}(Z_{t+1})+\gamma\left[E_t(Z_{t+1}-Z_{bt})^{\lambda}\right] \tag{6}$$

A second special case occurs when $\gamma \to \infty$. For large γ, maximizing (3) is the same as maximizing the 'stand-alone' tracking-error utility function

$$E_t(Z_{t+1}-Z_{bt})^{\lambda}-\phi\,\mathrm{var}_t(Z_{t+1}-Z_{bt}) \tag{7}$$

One might argue that, if the goal is to compare the behaviour of MV and TE types, then focusing on this exclusively tracking error based utility specification would be the best choice. The point here is that the TE objective (6) contains the stand-alone TE objective (7) as a special case. When γ is large, the maximizer of (6) approaches the maximizer of (7). By thoroughly checking the subsequent results for sensitivity to γ, the implications of the stand-alone objective are brought out as part of a full range of mean-variance/tracking error weightings.

2.3 *Benchmarks* Z_{bt}

So far, tracking error has been defined as a portfolio return's current performance relative to some other 'benchmark' level of performance. The benchmark in question has not yet been described in detail. This section specifies three distinct specifications of the benchmark, each of which may serve as a reference point against which real-world investors compare their own returns.

The first benchmark considered here is the last generation's return: $Z_{bt}=Z_t$. Experimental economics finds overwhelming evidence that human beings make many decisions by comparing alternatives to the status quo (Rabin, 1998). This gets at the idea of habituation, the notion that we 2
tend to become accustomed to a certain rate of progress and judge future progress relative to that baseline rate.

The second benchmark to be considered is the average (historical) return $Z_{bt}=\sum_{\tau=1}^{t} Z_\tau/t$. Financial commentators frequently compare current performance with historic rates of return. The historical benchmark will, of course, be less volatile through time than the previous benchmark ($Z_{bt}=Z_t$).

Finally, a third benchmark that can be linked to stylised facts established in the behavioural economics literature is the recent return of one's peers: i.e. TE types may focus on the recent returns of MV types as a reference point, reflecting a social comparison (as in Abel, 1990), and formalized in this model by specifying $Z_{bt}=Z_t^{MV}$. This social comparison benchmark gets at the idea that decision-makers (even expert fund managers) judge outcomes by comparing their own performance to that of their competitors. This benchmark captures the idea that investors judge losses to be less painful when others also lose.

2.4 *TE types take more risks*

The term in the TE types' utility function that distinguishes them from MV types is $E_t(Z_{t+1}-Z_{bt})^{\lambda}$, where λ is a free parameter (that must be a positive, odd integer). When $\lambda=1$, tracking error considerations do nothing more than place additional weight on mean-seeking relative to risk aversion, and it is straightforward to show that TE types take more risk and accumulate more wealth. For $\lambda=3$, it is no longer obvious whether TE types or MV types will take more risk. TE types' demand for risk depends non-trivially on the benchmark and can be less than or greater than MV types' demand for risk. For $\lambda\geq 5$, the question of whether TE or MV types take more risk is no less ambiguous. As in the $\lambda=3$ case, the technique of simulation is required when $\lambda\geq 5$ in order to study the distribution of the TE types' demand function x_t^{TE} in comparison to the distribution of x_t^{MV}. Because the $\lambda\geq 5$ case is qualitatively similar to $\lambda=3$, and because increasing λ leads to effects that resemble the effects of increasing λ (which is explicitly analysed later on in this paper), only the cases $\lambda=1$ and $\lambda=3$ are presented here.

2.4.1 *The TE demand function when $\lambda=1$*

When $\lambda=1$, the TE objective can be written as

$$\begin{aligned} u_t^{TE}(Z_{t+1}) &= u_t^{MV}(Z_{t+1})+\gamma E_t(Z_{t+1}-Z_{bt}) \\ &= E_t(Z_{t+1})(1+\gamma)-\gamma Z_{bt}-\frac{\beta}{2}\mathrm{var}_t(Z_{t+1}) \end{aligned}$$

It is immediately clear that the linear tracking error specification places additional weight on return versus risk. It is therefore no surprise that individuals with such preferences take more risk, which can be seen directly from

$$x_t^{TE}=\frac{\left[R_t^{\rho}e^{\mu+\frac{\sigma^2}{2}}-S\right](1+\gamma)}{\beta R_t^{2\rho}e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)}=x_t^{MV}(1+\gamma)$$

(so long as $x_t^{TE}\in[\underline{x},\bar{x}]$), implying that

$$\left|x_t^{TE}\right| > \left|x_t^{MV}\right|$$

Result 1. *Under the specified returns process, investors who maximize a linear tracking error objective take risk each period than do mean-variance investors.*

2.4.2 *The TE demand function when $\lambda=3$*

The goal is to find a choice $\bar{x}_t^{TE}$ that maximizes

$$u_t^{TE}(Z_{t+1})=u_t^{MV}(Z_{t+1})+\gamma E_t\left[(Z_{t+1}-Z_{bt})^3\right]$$

In order to maximize this function, the expression $E_t[(Z_{t+1}-Z_{bt})^3]$ is developed further. Re-expressing the difference $Z_{t+1}-Z_{bt}$ as $(Z_{t+1}-E_tZ_{t+1})+(E_tZ_{t+1}-Z_{bt})$ yields the relationship

$$E_t\left[(Z_{t+1}-Z_{bt})^3\right]=E_t^-\left[(Z_{t+1}-E_tZ_{t+1})^3\right]+3E_t\left[(Z_{t+1}-E_tZ_{t+1})^2\right](E_tZ_{t+1}-Z_{bt}) \quad (8)$$

$+(E_tZ_{t+1}-Z_{bt})^3=$ third central moment $+3\text{var}_t(Z_{t+1})\times(E_t\text{tracking error})+(E_t\text{tracking error}).^3$

As before. interest centres on analysing how the tracking error term (re-expressed above) affects the TE types' demand for risk. To approach this question, the signs of each term in (8) are sought.

Turning first to the sign of $E_t[(Z_{t+1}-E_tZ_{t+1})^3]$, one may note that $Z_{t+1}-E_tZ_{t+1}=x_t^3(R_{t+1}-E_tR_{t+1})$ and, therefore, that the third moment of Z_t is proportional to the third moment of R_t:

$$E_t\left[(Z_{t+1}-E_tZ_{t+1})^3\right]=x_t^3E_t\left[(R_{t+1}-E_tR_{t+1})^3\right]$$

It is also known that R_{t+1} is log-normal and therefore skewed to the right, implying that the third moment of R_{t+1} is positive. Thus, for $x_t>0$, $E_t[(Z_{t+1}-E_tZ_{t+1})^3]>0$.

Tsiang (1972) shows that any risk averse individual with decreasing absolute risk aversion with respect to wealth (as is the case here) will, all things equal, prefer positively skewed distribution. This might, at first glance, lead one to think that the questions being raised here follow trivially from the skewness of the returns distribution. This is not the case, however, because skewness is only one of several components of the expected cubic of tracking error, as the decomposition in (8) makes clear. That decomposition also makes explicit the existence of the subjective tension introduced by the consideration of tracking error: tracking error simultaneously creates a new preference for skewness and a new motive for averting risk when $(E_tZ_{t+1}-Z_{bt})<0$. This last feature is reflected in the term $3\text{var}_t(Z_{t+1})(E_tZ_{t+1}-Z_{bt})$ in (8), where it is seen that TE types may be more or less sensitive to the variance of Z_{t+1} depending on whether they expect to be below or above the benchmark Z_{bt}.

Referring back to the right-hand side of (8), all that can be concluded about the right-hand side is that the first term is positive while the second two terms have the same sign as $(E_tZ_{t+1}-Z_{bt})<0$. The entire tracking error expression $E_t[(Z_{t+1}-Z_{bt})^3]$ may be either positive or negative in x_t. When $(E_tZ_{t+1}-Z_{bt})>0$, it follows that $E_t[(Z_{t+1}-Z_{bt})^3]>0$, although the converse is not true. In other words, when the TE type expects to beat the benchmark, the TE type's utility function ranks any risky portfolio higher than the MV type does. When the TE type expects to fall short of the benchmark, however, the TE type may very well be more cautious than the MV type (but not always).

Next, the TE type's utility function is maximized. Appendix A expresses the TE type's utility function as a third-degree polynomial in x_t (the TE type's time t percentage of wealth allocated to risk):

$$u_t^{\text{TE}}[Z_{t+1}(x_t)]=a_0+a_1x_t+a_2x_t^2+a_3x_t^3$$

The coefficients, which are solved for explicitly in Appendix A, are functions of the exogenous parameters (μ, σ, β, ρ, γ, S) and the predetermined benchmark Z_{bt}. Solving the first-order condition for x_t yields two critical points. It must be verified whether these critical points lie in the admissible interval, and whether they do in fact maximize utility or not (since the utility function is not guaranteed to be concave in x_t). Evaluating the objective function at the boundary points as well as at the admissible critical points, the TE type's optimal response can be determined. Since the cubic polynomial has no upper bound, there will be parameterizations where the MV type plays a strategy on the boundary of the feasible interval $[\underline{x}, \bar{x}]$. (In such cases, the MV type may or may not choose an interior point.) The procedure for finding a maximizer for u_t^{TE} described above is formalized as follows.

Define r_1 and r_2 to be the two roots of

$$a_1 + 2a_2 x_t + 3a_3 x_t^2 = 0 \tag{9}$$

Then define

$$r_1^* = \begin{cases} r_1 & \text{if } r_1 \in [\underline{x}, \bar{x}] \\ \underline{x} & \text{otherwise} \end{cases} \quad \text{and} \quad r_2^* = \begin{cases} r_2, & \text{if } r_2 \in [\underline{x}, \bar{x}] \\ \underline{x} & \text{otherwise} \end{cases}$$

Finally,

$$x_t^{\mathrm{TE}} = \operatorname{argmax}_{x \in \{\underline{x}, r_1^*, r_2^*, \bar{x}\}} u_t^{\mathrm{TE}}[Z_{t+1}(x)]$$

Based on this behavioural equation, one can try to examine the effect of tracking error concern, γ, on the optimal percentage of wealth invested in risk x_t^{TE} (where the 'prime' mark indicates differentiation with respect to γ, $M \equiv E_t R_{t+1} = R_t^{\rho} e^{\mu + \frac{1}{2}\sigma^2}$, and $V \equiv \mathrm{var}_t(R_{t+1}) = R_t^{2\rho} e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$):

$$\frac{dx_t^{\mathrm{TE}}}{d\gamma} = -\frac{a_1' + 2a_2' x_t^{\mathrm{TE}} + 3a_3'\left(x_t^{\mathrm{TE}}\right)^2}{6a_3 + 2a_2}$$

$$= -\frac{3(M-S)(S-Z_{\mathrm{b}t})^2 + 6\left[V - (M-S)^2\right](S - Z_{\mathrm{b}t})x + 3\left[R_t^{3\rho} e^{3\mu + \frac{9}{2}\sigma^2} - 3VM - M^3 + 3V(M-S) + (M-S)^3\right]x^2}{6\gamma\left[R_t^{3\rho} e^{3\mu + \frac{9}{2}\sigma^2} - 3VM - M^3 + 3V(M-S) + (M-S)^3\right] - \beta V + 6\gamma\left[V + (M-S)^2\right](S - Z_{\mathrm{b}t})}.$$

The sign of the expression above is indeterminate, depending on the relative magnitudes of M and V (which are exogenously given) and the sign of $(S - Z_{\mathrm{b}t})$. This means that in any given period, tracking error can lead to more or less risk-taking, depending on the size of the most recent realization of $Z_{\mathrm{b}t}$ relative to S.

Result 2. *For investors with (third-order) non-linear tracking error preferences, risk-taking is non-monotonic in the intensity of tracking-error considerations, γ: i.e.*

$$\gamma\uparrow \Rightarrow x_t^{TE} \uparrow \text{ or } \downarrow$$

With the effect of TE considerations on risk-taking ambiguous in the non-linear case, the technique of simulation may be used to study the average effect and other aspects of the probability distribution associated with the difference between TE and MV demand, $D_{ts} \equiv x_t^{\mathrm{TE}(s)} - x_t^{\mathrm{MV}(s)}$. The primary question is whether there is a relationship between tracking error and risk-taking that holds on average, even though no such relationship holds for a particular realization of D_{ts}. For each of a number of different parameterizations, 1000 realizations of a 100 period demand sequence $\left\{\left[x_t^{\mathrm{MV}}, x_t^{\mathrm{TE}}\right]\right\}_1^{100}$ were computed. Figure 3 shows a typical empirical distribution of 100 000 simulated differences in levels of risk-taking. For most parameterizations, TE types always take more risk, as in Figure 3. But for other parameter values, there are individual realizations of D_{ts} that are negative for a particular time and run, although the time average $\frac{1}{100}\sum_{t=1}^{100} D_{ts}$ is consistently positive across realizations s. 3

Table 1 presents the average difference

$$\bar{D} \equiv 10^{-5} \sum_{s=1}^{1000} \sum_{t=1}^{100} \left(x_t^{\mathrm{TE}(s)} - x_t^{\mathrm{MV}(s)}\right)$$

and its associated t-statistics over a range of parameter values. The t-statistics in Table 1 correspond to a test of the null hypothesis that the average difference in risk-taking is zero, H_0: $E_T\left(x_t^{TE} - x_t^{MV}\right) = 0$. The alternative hypothesis of interest here is the idea that TE types take more risk on average, i.e. H_1: $E_T\left(x_t^{TE} - x_t^{MV}\right) > 0$. Judging from the overwhelmingly large, positive *t*-statistics in Table 1, there is broad support for the notion that TE types take more risk (on average) than MV types do, even though both types share the same degree of risk aversion β.

Result 3. *Investors with (third-order) non-linear tracking error preferences take more risk on average than MV types do: i.e.* $\mathrm{Ex}^{TE} > \mathrm{Ex}^{MV}$ *holds for many parameter values.*

Table 1 reveals a number of other noteworthy qualitative patterns. For instance, the difference in levels of risk-taking across types is more significant when the risk premium $e^{\mu + \frac{\sigma^2}{2}} - S$ is higher, when volatility σ^2 is lower, and when both types are less risk averse (small β). The persistence of returns ρ has non-monotonic effects on average risk-taking. Also, the borrowing constraints ($\underline{x}$, $\bar{x}$) and the length of the time sequence (T) have little effect on the relative level of risk-taking, at least for the parameter values considered. Perhaps most important, Table 1 confirms the link between tracking error and risk-taking, evidenced by the increasing differences in risk-taking that go together with increasing values of the tracking error parameter γ.

Table 1 also allows one to compare relations between risk-taking and parameter values across the three different benchmarks that were specified earlier. The differences appear to be robust across the three specifications, although they are less pronounced for the second benchmark (historical average). The next section analyses the same simulation data using the distinct, although related, criterion of accumulated wealth.

2.5 *Distribution of long-run wealth accumulations*

In this section, the long-run wealth accumulations of TE and MV types are compared. In particular, this section computes a simulated probability distribution for the TE type's share of aggregate accumulated wealth 100 generations hence. As before, the quantities being compared arise from decisions taken in isolation, i.e. decisions in an environment where MV and TE types do not interact. The two types face identical realizations from a single innovation process $\{\varepsilon_t\}_{-}^{100}$, however. In a subsequent section, identical comparisons are made for the equilibrium model in which both types of investors do interact. Also, it is important to realize that there is no forward-looking behaviour reflected in the simulations presented here, since we are interpreting the time increment to be one generation and assuming that each generation cares mostly about itself.

After simulating wealth accumulations over 100 periods separately for MV and TE demand functions, the empirical distribution of the TE type's share of accumulated wealth is tabulated, where the variable TESHARE is defined as

$$\mathrm{TESHARE} \equiv \frac{W_{100}^{TE}}{W_{100}^{TE} + W_{100}^{MV}}$$

Figure 1 shows the empirical probability density function of TESHARE based on 1000 simulated 100 generation sequences. TE types (with third-order tracking error preferences) always accumulate more than half the aggregate wealth accumulated after 100 generations. Table 2 4

Table 1. Difference in risk-taking $D = (x_t^{TE} - x_t^{MV})$ for different parameter values [non-equilibrium case].

	Parameter value	$\bar{D}$ *benchmark 1*	*t*	$\bar{D}$ *benchmark 2*	*t*	$\bar{D}$ *benchmark 3*	*t*
μ	0.0200	0.0000	0.9312	0.0000	1.9783	0.0000	0.9313
	0.0400	0.0010	4.9037	0.0005	14.1306	0.0010	4.9231
	0.0600	0.0067	7.7793	0.0036	25.5283	0.0067	7.8719
	0.0800	0.0216	9.0437	0.0109	31.5326	0.0216	9.2420
	0.1000	0.0513	10.0438	0.0247	32.1101	0.0517	10.1774
σ^2	0.0016	0.0310	9.1088	0.0157	30.5845	0.0309	9.2792
	0.0049	0.0036	7.6030	0.0019	23.9122	0.0036	7.6787
	0.0100	0.0011	6.7201	0.0006	17.7393	0.0011	6.7634
	0.0169	0.0005	6.3922	0.0003	14.2833	0.0005	6.4230
	0.2500	0.0001	10.9813	0.0001	8.0382	0.0001	11.0156
β	20.0000	0.0434	17.2864	0.0456	34.5175	0.0440	17.6064
	40.0000	0.0102	8.7465	0.0053	28.7721	0.0102	8.8708
	70.0000	0.0018	9.0580	0.0009	29.2628	0.0018	9.0967
	110.0000	0.0005	9.2145	0.0002	29.9465	0.0005	9.2303
	160.0000	0.0002	9.3559	0.0001	18.1602	0.0002	9.3635
ρ	0.0200	0.0000	0.9312	0.0000	1.9783	0.0000	0.9313
	0.1000	0.0152	12.9210	0.0108	17.8099	0.0153	13.3502
	0.3000	0.0292	30.3763	0.0303	11.3256	0.0304	30.0129
	0.5000	0.0223	7.0718	0.0131	6.1379	0.0231	7.0184
	0.7000	0.0026	1.7045	0.0027	1.6017	0.0026	1.6996
	0.9000	0.0001	0.3360	0.0001	0.3245	0.0001	0.3353
A	0.0100	0.0127	8.6640	0.0065	28.4548	0.0127	8.8088
	0.1000	0.0127	8.6640	0.0065	28.4548	0.0127	8.8088
	1.0000	0.0127	8.6640	0.0065	28.4548	0.0127	8.8088
	2.0000	0.0127	8.6640	0.0065	28.4548	0.0127	8.8088
	3.0000	0.0127	8.6640	0.0065	28.4548	0.0127	8.8088
T	10.0000	0.0121	2.8274	0.0075	4.6374	0.0121	2.8679
	50.0000	0.0127	6.1566	0.0068	17.1224	0.0127	6.2559
	75.0000	0.0128	7.4905	0.0066	24.5061	0.0128	7.6153
	100.0000	0.0127	8.4861	0.0064	29.6067	0.0126	8.6296
	200.0000	0.0127	12.3530	0.0067	56.6599	0.0127	12.5621
γ	0.1000	0.0002	9.2734	0.0001	29.7541	0.0002	9.2761
	1.0000	0.0024	9.1718	0.0013	29.5252	0.0024	9.1994
	10.0000	0.0288	7.7924	0.0135	26.9587	0.0286	8.1321
	50.0000	0.2403	10.9503	0.0925	11.4001	0.2712	11.8945
	100.0000	0.3318	16.0389	0.3431	15.5648	0.3946	20.3437

There are 100 000 realizations of D for each parameter value. When not specified otherwise, the other parameters are centred at the values $\mu = 0.07$, $\sigma = 0.10$, $\beta = 37.31$, $\gamma = 5$, $\rho = 0$, $A = 0$, $T = 100$, and $S = 1.02$. Benchmark 1 is the most recent own return, $Z_{bt} = Z_t$. Benchmark 2 is the historical benchmark $Z_{bt} = \sum_{\tau=1}^{t} Z_\tau / t$. And Benchmark 3 is the MV type's most recent return, $Z_{bt} = Z_t^{MV}$.

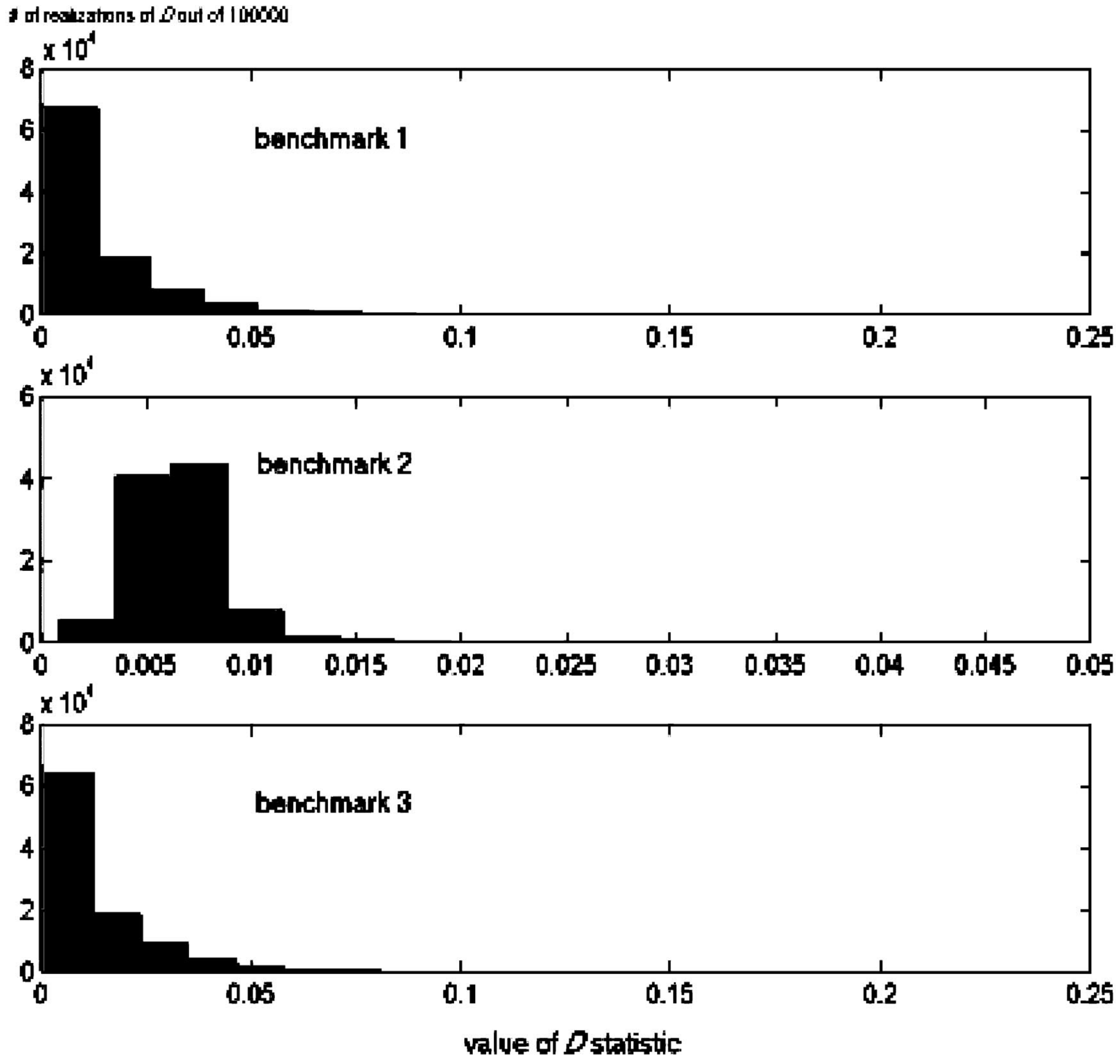

Fig. 1. Empirical pdf of $D=\left(x_t^{\mathrm{TE}}-x_t^{\mathrm{MV}}\right)$ [non-equilibrium case].

presents average TESHARE statistics for a range of parameter values, with t-statistics corresponding to the null hypothesis of H_0: TESHARE$=$0.5. The relevant alternative hypothesis in this context is H_1: TESHARE $>$0.5, which is favoured by the test statistics at high levels of significance.

Table 2 shows that the TE types' expected share of long-run wealth (average TESHARE) is an increasing function of the expected risky return (μ), decreasing in risk (σ), and decreasing in the degree of risk-aversion (β). Table 2 also shows that average TESHARE is non-monotonic in the persistence of the returns process (ρ) and increasing in the time horizon (T). Also evident from Table 2 is the

Table 2. TE-share (TESHARE) of terminal aggregate wealth for different parameter values [non-equilibrium case].

	Parameter value	$\overline{\text{TESHARE}}$ *benchmark 1*	t	$\overline{\text{TESHARE}}$ *benchmark 2*	t	$\overline{\text{TESHARE}}$ *benchmark 3*	t
μ	0.0200	0.5000	0.5498	0.5000	0.9849	0.5000	0.5497
	0.0400	0.5005	21.0197	0.5003	24.9248	0.5005	21.0230
	0.0600	0.5068	54.1107	0.5036	52.1027	0.5068	54.8226
	0.0800	0.5322	85.8802	0.5163	68.9802	0.5322	90.3293
	0.1000	0.5982	103.1811	0.5479	89.1831	0.5989	115.4388
σ^2	0.0016	0.5377	85.7085	0.5193	73.8142	0.5377	90.4957
	0.0049	0.5047	51.7619	0.5025	46.3474	0.5047	52.1945
	0.0100	0.5015	34.0337	0.5008	33.4754	0.5015	34.1282
	0.0169	0.5007	26.1809	0.5004	27.0334	0.5007	26.2420
	0.2500	0.5005	23.4830	0.5003	19.5241	0.5005	23.5060
β	20.0000	0.5520	107.5741	0.5544	68.5804	0.5527	111.2043
	40.0000	0.5128	75.7858	0.5066	58.5771	0.5128	77.7822
	70.0000	0.5023	77.9201	0.5012	55.9656	0.5023	78.5527
	110.0000	0.5006	79.0791	0.5003	56.2639	0.5006	79.3388
	160.0000	0.5002	80.9339	0.5001	60.7681	0.5002	81.0602
ρ	0.1000	0.5206	108.5185	0.5159	67.4330	0.5208	113.1275
	0.3000	0.5551	135.9365	0.5638	73.1963	0.5574	142.7373
	0.5000	0.5479	62.0716	0.5288	61.9260	0.5496	61.0096
	0.7000	0.5056	16.9129	0.5058	16.1291	0.5057	16.7995
	0.9000	0.5003	2.8003	0.5002	2.7603	0.5003	2.7971
A	0.0100	0.5160	75.1440	0.5082	58.8476	0.5159	77.4604
	0.1000	0.5160	75.1440	0.5082	58.8476	0.5159	77.4604
	1.0000	0.5160	75.1440	0.5082	58.8476	0.5159	77.4604
	2.0000	0.5160	75.1440	0.5082	58.8476	0.5159	77.4604
	3.0000	0.5160	75.1440	0.5082	58.8476	0.5159	77.4604
T	10.0000	0.5015	8.2500	0.5009	9.1618	0.5015	8.3777
	50.0000	0.5079	37.6028	0.5043	37.7968	0.5079	38.4271
	75.0000	0.5120	59.2959	0.5062	51.4446	0.5119	61.0070
	100.0000	0.5160	77.6635	0.5082	61.7150	0.5160	79.8204
	200.0000	0.5313	142.3731	0.5167	124.3698	0.5313	145.2752
γ	0.1000	0.5003	80.4383	0.5002	60.1162	0.5003	80.4832
	1.0000	0.5030	79.5929	0.5016	59.8865	0.5030	80.0419
	10.0000	0.5360	67.4758	0.5168	57.5015	0.5357	72.2599
	50.0000	0.7643	97.9794	0.6127	45.0906	0.7905	128.9938
	100.0000	0.8348	143.4286	0.8410	120.4850	0.8721	188.2763

The t-statistic here is TESHARE-0.5. There are 1000 realizations of TESHARE for each parameter value. When not specified otherwise, the other parameters are centred at the values $\mu = 0.07$, $\sigma = 0.10$, $\beta = 37.31$, $\gamma = 5$, $\rho = 0$, $A = 0$, $T = 100$, and $S = 1.02$. Benchmark 1 is the most recent own return, $Z_{bt} = Z_t$. Benchmark 2 is the historical benchmark $Z_{bt} = \sum_{\tau=1}^{t} Z_r/t$. And Benchmark 3 is the MV type's most recent return, $Z_{bt} = Z_t^{\text{MV}}$.

dramatic role the tracking error parameter γ plays in the accumulation of wealth. The more intensely decision-makers care about tracking error, the more they take risk and dominate MV types by the measure of accumulated wealth.

Result 4. *Investors with (third-order) non-linear tracking-error preferences accumulate more wealth than identically risk-averse MV types do, across the full range of parameter values and benchmark specifications.*

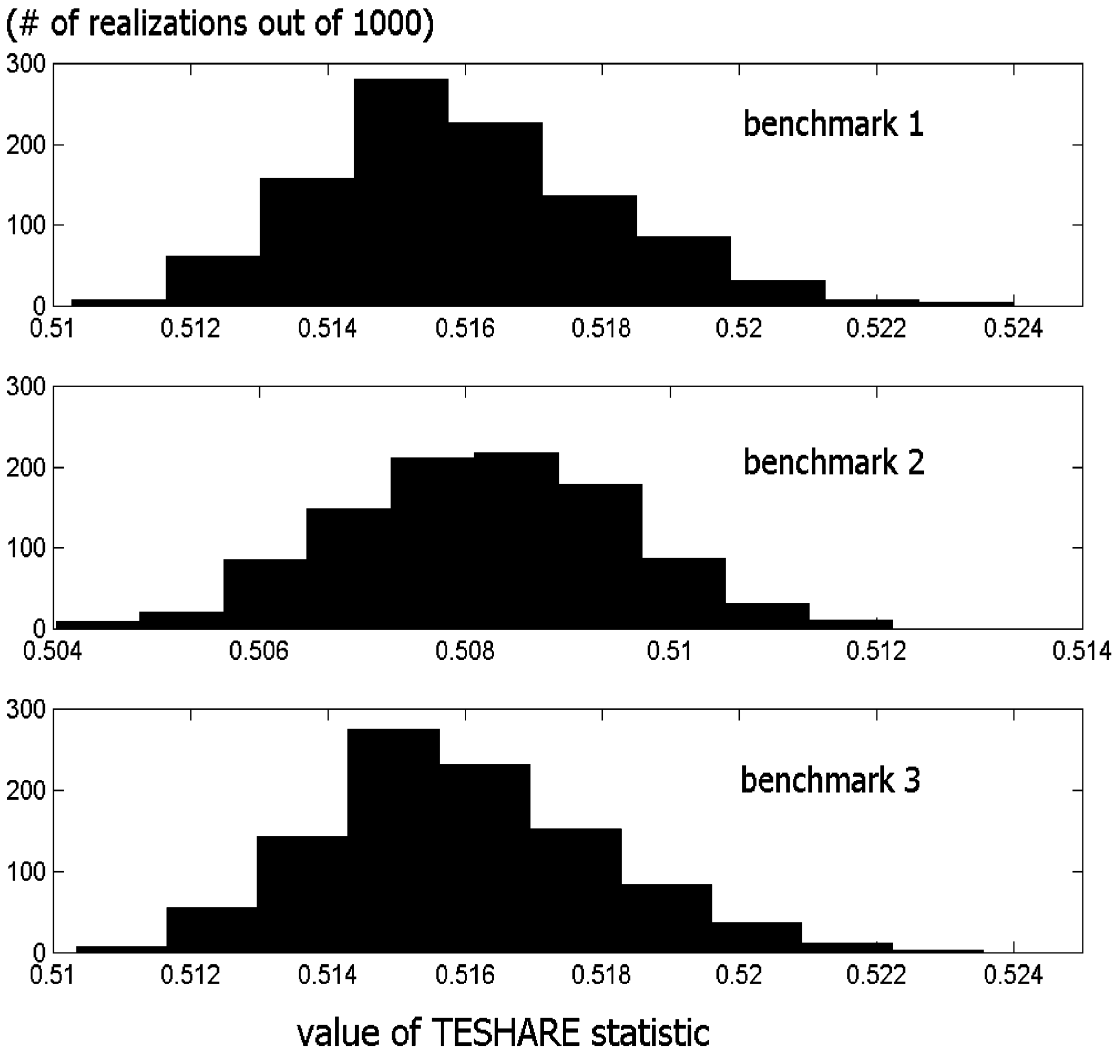

Fig. 2. Empirical pdf of TESHARE [non-equilibrium case].

Figure 3 depicts the average share of own wealth that TE types allocate to risk at each point in time, along with 80% confidence bands. At least 80% of the time, the TE types put more than half their wealth into risk, whereas the MV types (not pictured) make choices centred around the 50% level, by construction. The risk aversion parameter β is calibrated so that when returns are a random walk (asset prices follow geometric Brownian motion, i.e. $\rho = 0$), MV types follow the popular rule of thumb 'half in stocks, half in bonds'.

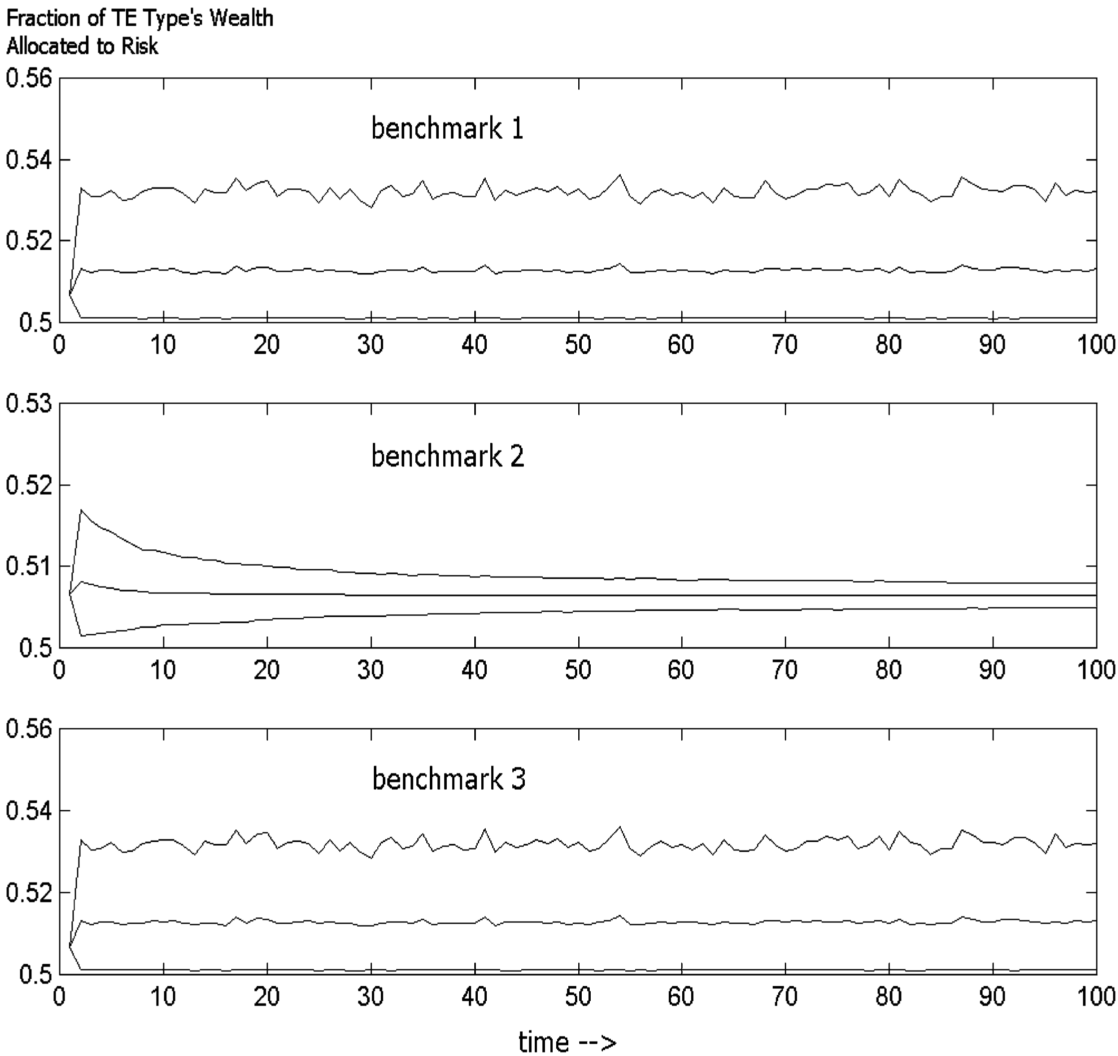

Fig. 3. TE type's fraction of own wealth allocated to risk (average x_t^{TE} with 80% confidence bands) [non-equilibrium case].

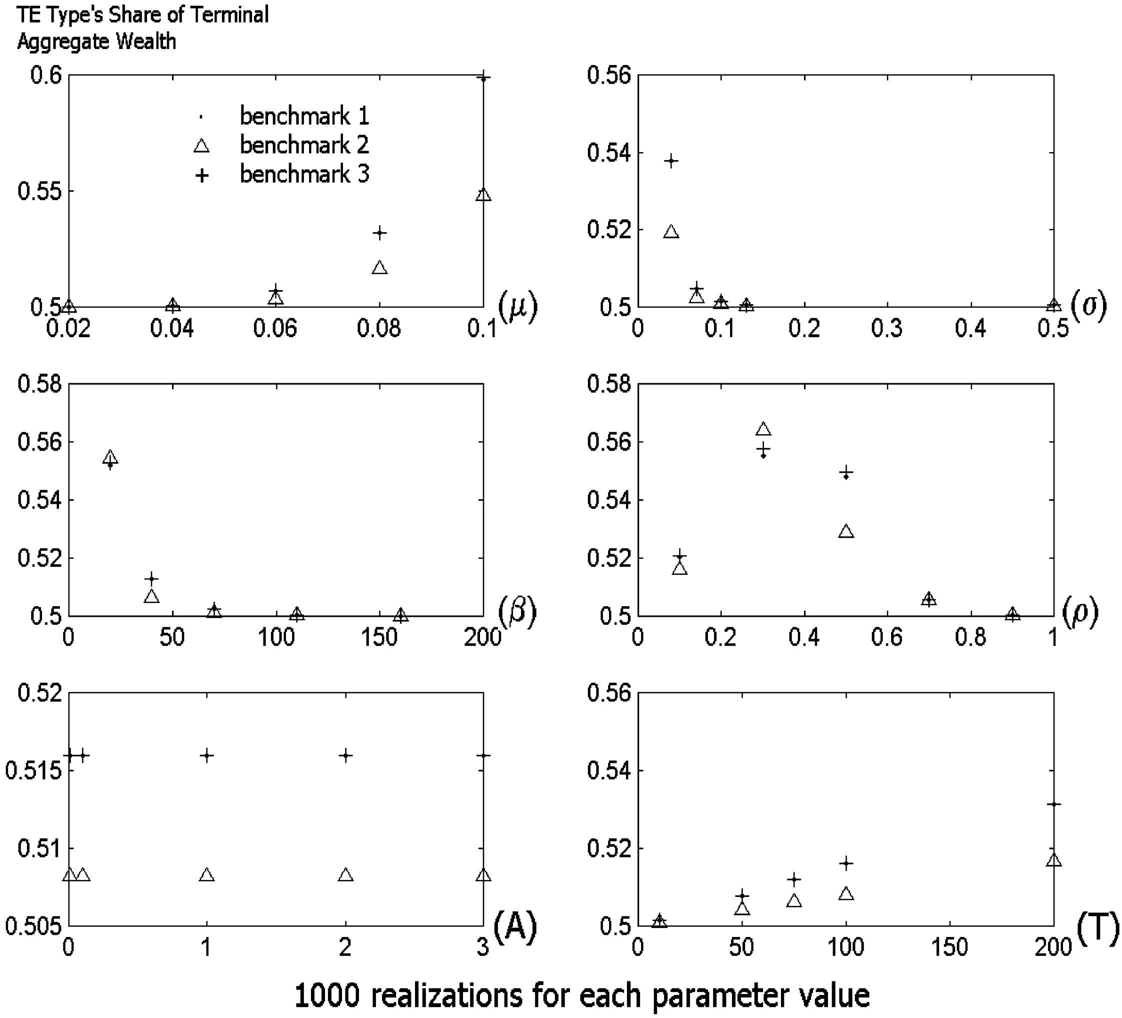

Fig. 4. Response of TESHARE to exogenous changes [non-equilibrium case].

Figure 3 shows how the TE type's share of terminal aggregate wealth changes in response to changes in parameter values. A number of the phenomena discussed earlier are easy to see in Figure 3: average TESHARE is increasing in the risk premium, decreasing in volatility, and decreasing in risk-aversion. 5

3. Endogenizing expectations in an equilibrium where TE and MV types interact

Having investigated the distribution of accumulated wealth under decision-making in isolation, this section now undertakes to study a similar set of comparisons when the two types interact. Instead of

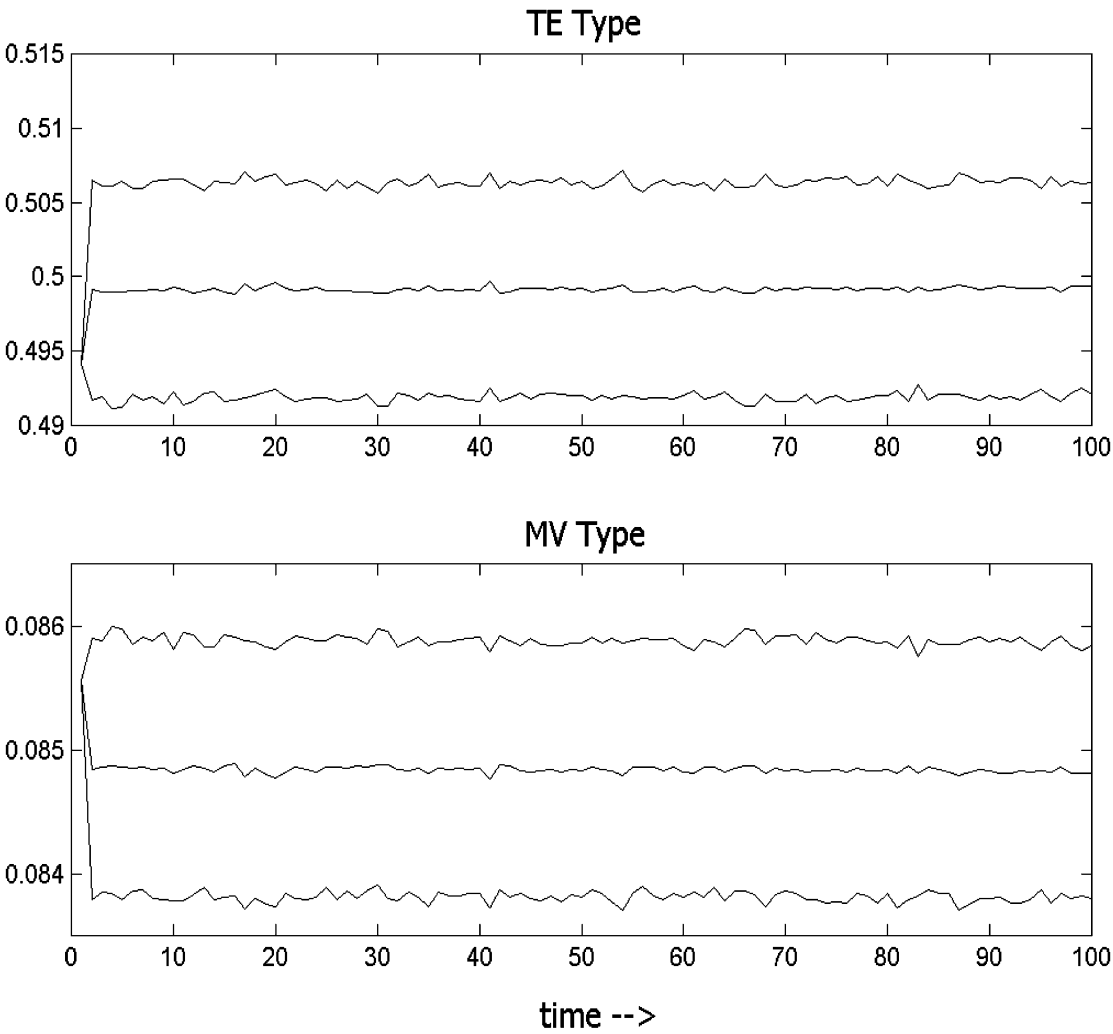

Fig. 5. TE and MV types' share of own wealth allocated to risk (average x_t^{TE} and average x_t^{MV} with 80% confidence bands), Benchmark 1 [equilibrium case].

analysing the accumulated wealth of both types in isolation as before, the actions and ultimate fortunes of both types are considered to co-exist in an environment where risky returns reflect the decisions of both types. The goal of this section therefore is to analyse TE decision-making when the actions of TE and MV types affect the returns process itself.

When co-existing in a single environment, the decisions x_t^{MV} and x_t^{TE} must be tied together by an economy-wide constraint. Such a constraint ought to reflect the inherent scarcity of risky opportunities, and also ought to generalize the earlier isolation model so that each type's individual results

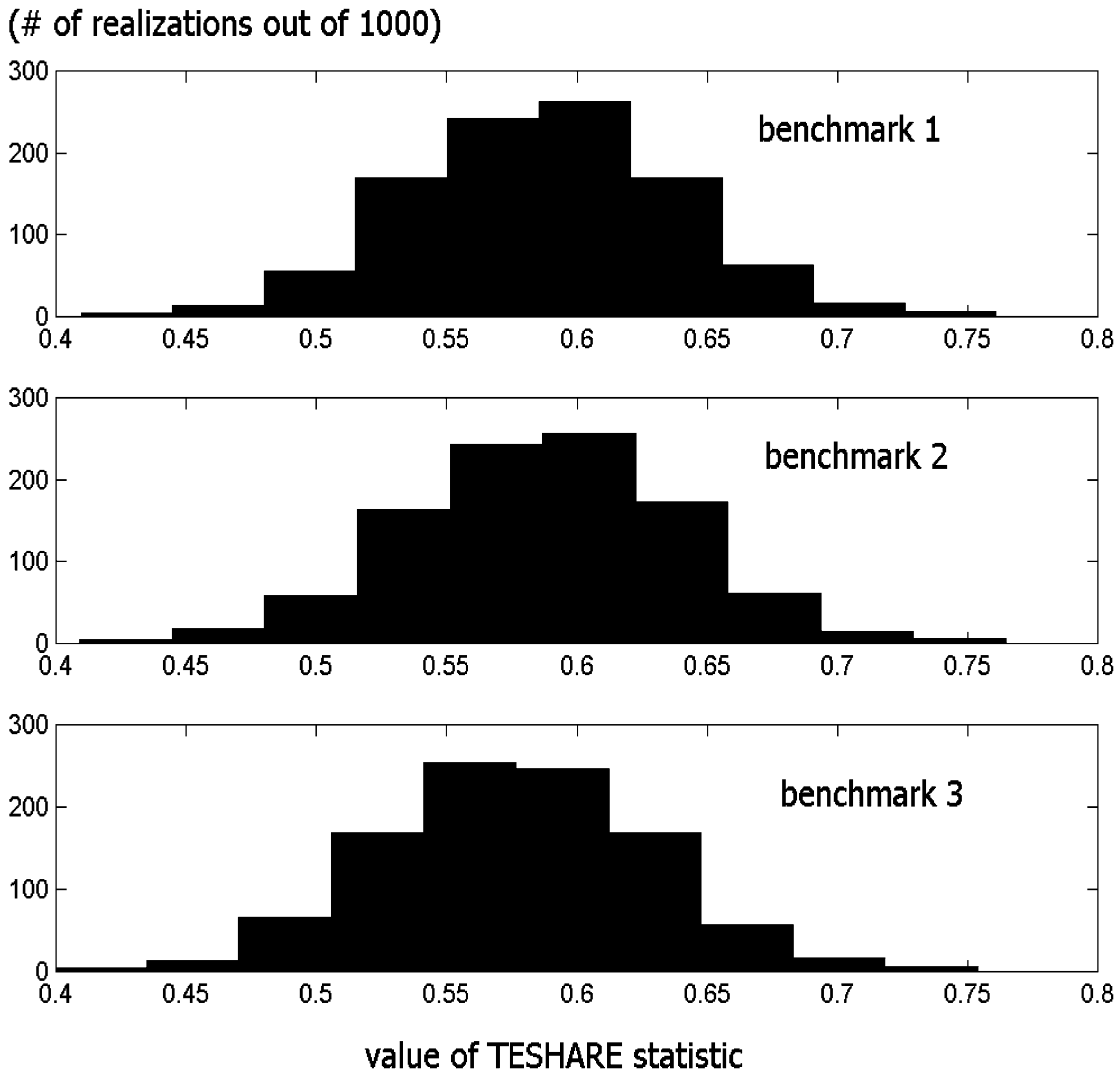

Fig. 6. Empirical pdf of TESHARE [equilibrium case].

may be recovered if the other type opts out of the market entirely. The latter requirement simply means that when x_t^{TE} is set to zero, the equilibrium model's optimal x_t^{MV} ought to be the same as it was in isolation, because the TE type is not pursuing any risk at all and the MV types are therefore making investment decisions in isolation once again.

While the second requirement that equilibrium effects disappear when parameters are such that $x_t^{\mathrm{TE}}=0$ or $x_t^{\mathrm{MV}}=0$ is an important technical condition, the earlier requirement that there be a supply constraint on risky opportunities is a substantive one deserving further explanation. The key idea is this: the more risk-taking that occurs in the aggregate, the lower is the return on risk-taking.

In any given period, actual firms raise capital by, in essence, submitting a menu of risky projects to wealth holders who in turn supply capital. These lists of risky projects are finite and span a range of levels of quality, summarized as return and risk characteristics. Assuming that the risk and return characteristics of projects is known and that the best projects are taken first, that leaves only inferior projects to be added in periods when there is high demand for risk. Thus, more risk-taking implies that lower quality opportunities are being pursued, i.e. the average quality of risk is declining in the level of aggregate risk-taking.

The recent technology bubble in the US stock market illustrates the point. Because of a higher-than-usual demand for the opportunity to be exposed to the risk of technology firms, lower quality projects were funded by investors. As further illustration in a non-financial context, one might imagine a hunter-gatherer society where hunting is risky and gathering is safe. When more hunting occurs, the expected quantity of prey goes down, i.e. the return on risk-taking falls. Thinking about risky opportunities over a time frame where it is reasonable to assume they are in fixed supply leads to an analysis which is completely analogous to David Ricardo's (1971) classic example of progressively less and less productive agricultural enterprises being undertaken.

One important potential complication of this analysis is that increased risk-taking might lead to innovations that increase the expected return of those who pursue risk. If so, increased risk-taking would have an ambiguous effect on average return due to the endogeneity of risk-taking and average returns. Undoubtedly, there are simultaneous causal channels from return to risk-taking and risk-taking to return. In this paper, however, we posit that the 'return causes risk-taking' channel is much speedier and more transparent than the reverse. Thus, a fixed supply of risky opportunities is assumed to constrain each generation.

The idea that average return on risk is decreasing in the level of aggregate risk is formalized in the following aggregate supply constraint:

$$R_{t+1} = e^{\varepsilon_{t+1}}(1 - x_t^{\mathrm{MV}} x_t^{\mathrm{TE}}) \tag{10}$$

where x_t^{MV} and $x_t^{\mathrm{TE}} \in [0,1]$, and $\in_{t+1} \sim \mathrm{N}(\mu, \sigma^2)\forall t$. When everyone pursues risk exclusively ($x_t^{\mathrm{MV}} = x_t^{\mathrm{TE}} = 1$), gross return is zero. That means both groups lose their entire accumulation of wealth. (Recall that $Z_{t+1} = \frac{W_{t+1}}{W_t} = [x_t R_{t+1} + (1 - x_t)S]$. If $x_t^{\mathrm{MV}} = x_t^{\mathrm{TE}} = 1$, (10) implies $R_{t+1} = 0$ which, in turn, implies that $W_{t+1} = 0$ for both types.)

One can also see from (10) that, if either type takes no risk at all ($x_t^{\mathrm{MV}} = 0$ or $x_t^{\mathrm{TE}} = 0$), then the other type will face the risky returns process $R_{t+1} = e^{\varepsilon_{t+1}}$. This is identical to the returns process analysed in the earlier section. By encompassing the earlier (isolated) decision rules as a special case within the general equilibrium framework analysed here, a proper comparison of tracking error effects in partial versus full equilibrium is possible. Imposing the constraint (10) closes the model in the sense that decisions and expectations $\left[x_t^{\mathrm{MV}}, x_t^{\mathrm{TE}}, E_t R_{t+1}\right]$ are now endogenous functions of the exogenous parameters $\theta \equiv [\mu, \sigma, S, \beta, \gamma]$.

The 'price' in the equilibrium model is simply the expected return

$$p \equiv E_t R_{t+1}$$

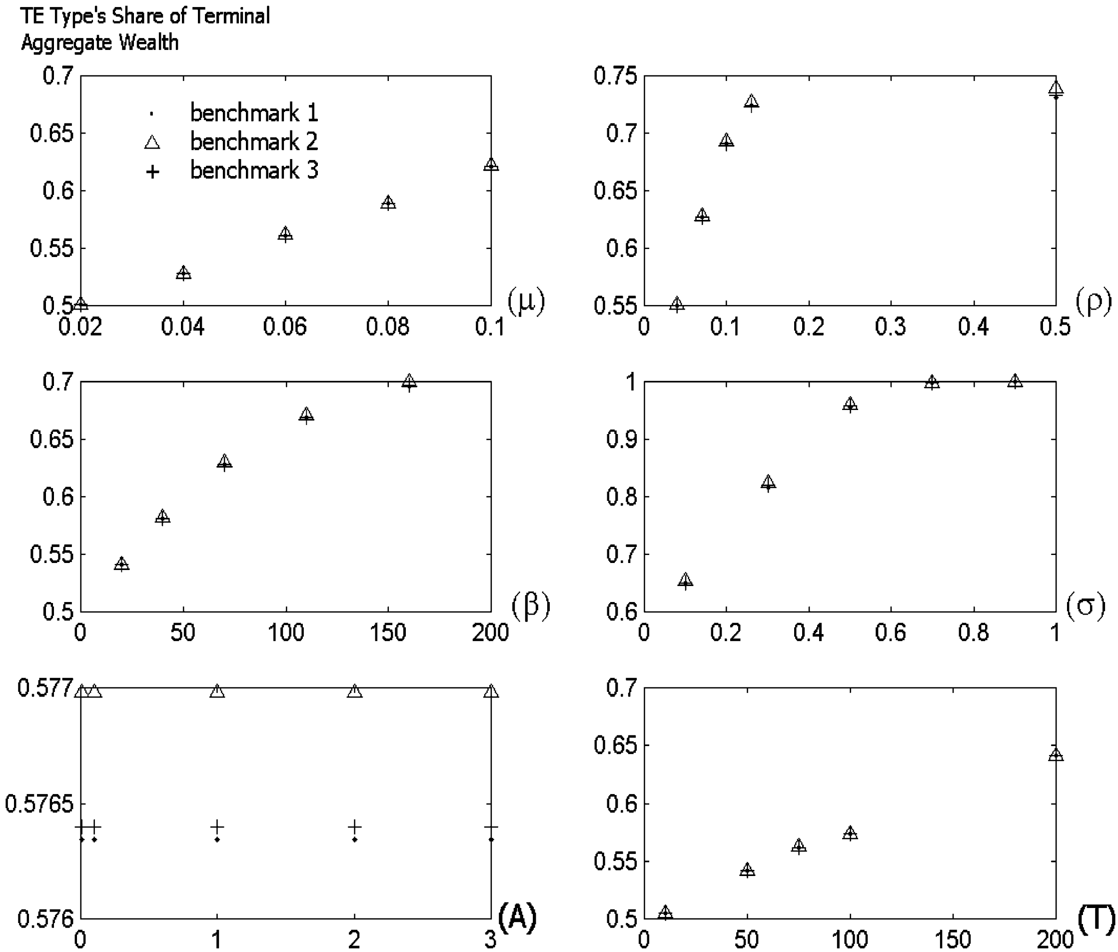

Fig. 7. Response of TESHARE to exogenous changes [equilibrium case].

To simplify notation for the remainder of the paper, x_t^{MV} and x_t^{TE} are rewritten more simply as

$$x = x_t^{\mathrm{MV}} \text{ and } y = x_t^{\mathrm{TE}} \tag{11}$$

With this convention, the resource constraint (10) becomes $R_{t+1} = e^{\varepsilon_{t+1}}(1 - xy)$. Taking expectations, the constraint can be written in terms of price:

$$p = m(1 - xy), m \equiv e^{\mu + \frac{\sigma^2}{2}}$$

Variance, in addition to price, is also endogenously determined:

$$\mathrm{var}_t R_{t+1} = p^2 v, \text{with } v \equiv e^{\sigma^2} - 1$$

At this point, for purposes of tractability, we revert to the linear tracking error utility function $u_t^{\mathrm{TE}} = u_t^{\mathrm{MV}} + \gamma E_t\left(\frac{Z_{t+1}}{Z_{bt}} - 1\right)$. Of course, in reverting to the linear tracking error specification for the equilibrium model, it follows trivially that the TE types take more risk. However, the increased risk-seeking of TE types gives rise to a newly accounted for social cost, reflected in the constraint (10), the net effect of which is not at all obvious. The key question to be addressed by the equilibrium model is whether the presence of TE types elevates or reduces aggregate wealth. The propensity of TE types to take on additional risk leads them to accumulate more wealth than MV types. That much is clear. But it remains to determine whether TE types help the market accumulate more wealth over time or wind up depriving society of wealth it would have enjoyed under homogeneous MV preferences. Thus, this section is intended to compare equilibrium aggregate wealth with and without tracking error types. This section also compares aggregate wealth in the heterogenous preference equilibrium model against the maximum possible aggregate wealth achievable by a benevolent planner who picks demand rules in accordance with individual preferences and in a manner that takes into account the social costs of pursuing risk.[2]

Seeking to endogenize price p, one begins by writing both utility functions in terms of price:

$$u_t^{\mathrm{MV}}(x) = S + (p - S)x - \frac{\beta}{2} p^2 v x^2$$
$$u_t^{\mathrm{TE}}(y) = S\left(1 + \frac{\gamma}{Z_{bt}}\right) - \gamma + (p - S)\left(1 + \frac{\gamma}{Z_{bt}}\right)y - \frac{\beta}{2} p^2 v y^2$$

Just as in the Walrasian model, agents maximize their objectives taking p as given, not considering how their own actions affect price. For interior choices of x and y,

$$x = \frac{p - S}{\beta v p^2} \text{ and } y = \frac{(p - S)\left(1 + \frac{\gamma}{Z_{bt}}\right)}{\beta v p^2}$$

Imposing the supply constraint, equilibrium p must satisfy the equality

$$p = m(1 - xy) = m\left[1 - \frac{(p - S)^2\left(1 + \frac{\gamma}{Z_{bt}}\right)}{\beta^2 v^2 p^4}\right]$$

Thus, an equilibrium expected return p is defined as a solution to the auxiliary equation

$$h(p) \equiv \beta^2 v^2 p^4\left(\frac{p}{m} - 1\right) + \left(1 + \frac{\gamma}{Z_{bt}}\right)(p - S)^2 = 0$$

If $p < S$, there is no reason for risk averse decision-makers to invest anything in risk, i.e. $x = y = 0$, which cannot be an equilibrium. Therefore, one may assume $p \geq S$. Also, restricting x and y to [0, 1] (no borrowing allowed), the inequality $p \leq m$ must hold at any equilibrium, since $p = m(1 - xy)$ and $0 \leq xy \leq 1$. The admissible range of p is therefore $[S, m]$, an intuitive result stating that the equilibrium

[2]Regarding the tractability of using the third-order tracking error preference specification, it is actually quite tractable to compute prices that clear markets. Just as with the linear specification, one implicitly defines equilibrium price by using an auxiliary equation, which turns out to be a seventh-order polynomial equation in p. The primary technical challenge that arises relates to the fact that there typically exist a multiplicity of solutions. The tractability issue concerns the need to invoke new assumptions, or probabilistic rules, to decide among the candidate solutions. We were unable to find a theoretically motivated procedure of this sort. Therefore, the linear specification re-appears in this section. Happily, the simpler form continues to allow the equilibrium model to address some issues of substance, in spite of the fact that tracking error transparently forces TE types to always take on additional risk.

return on risk must fall somewhere between the safe activity's return (S) and the return on risk enjoyed by a single person pursuing risk in isolation (m).

The existence of an equilibrium price $p^* \in [S, m]$ satisfying $h(p^*) = 0$ must be verified. Two inequalities help prove existence. Evaluating the auxiliary function $h(p)$ at the endpoints of the interval $[S, m]$,

$$h(S) = \beta^2 v^2 S^4 \left(\frac{S}{m} - 1\right) < 0, \text{and}\, h(m) = \left(1 + \frac{\gamma}{Z_{bt}}\right)(m - S)^2 > 0 \tag{12}$$

where the inequalities both follow from $S < m$. By the continuity of h and an application of the intermediate value theorem, existence is established.

The next question to address is how many solutions to $h(p) = 0$ exist in the interval $[S, m]$. In order to guarantee uniqueness, it must be shown that $h(p)$ crosses the x-axis just once on $[S, m]$. This is guaranteed if h is strictly increasing on $[S, m]$, i.e. if

$$h'(p) = 5\beta^2 v^2 p^3 \left(5\frac{p}{m} - 4\right) + 2\left(p - \frac{\gamma}{Z_{bt}}\right) > 0 \tag{13}$$

Condition (13) obviously holds at $p = m$. And the condition holds everywhere so long as the exogenous parameters satisfy

$$\log\left(\frac{5}{4}S\right) > \mu + \sigma^2/2 \tag{14}$$

which is easily satisfied for all parameterizations in a large neighbourhood of estimates corresponding to these parameters (as measured by the historical stock market data from the USA). For example, at $\mu = 0.08$, $\sigma = 0.20$, and $S = 1.02$, the inequality is comfortably satisfied: $\log(1.25 \times 1.02) = 0.24 > 0.08 + 0.20^2/2$. Thus, the parameter restriction (14) is imposed throughout the remainder of this paper, guaranteeing the uniqueness of equilibrium price p.

The admissible parameter space, denoted Ω, is defined by the following inequalities:

$$\mu \geq 0,\ \sigma^2 \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ e^{\mu + \frac{\sigma^2}{2}} \geq S \geq 1, \text{ and } \log\left(\frac{5}{4}S\right) > \mu + \sigma^2/2$$

Result 5. *When $\theta \equiv [\mu, \sigma^2, \beta, \gamma, S] \in \Omega$, the economy defined by the sequence of agents $\{MV_t, TE_t\}_{t=1}^T$, the sequence of preferences $\{MV_t, TE_t\}_{t=1}^T$, and the sequence of pre-determined endowments $\{W_t^{MV}, W_t^{TE}\}_{t=0}^{T-1}$ has a unique equilibrium price sequence (expected return se-quence) $\{p_t\}_{t=1}^T$.*

3.1 *Comparative statics*

By implicitly differentiating the auxiliary equation which defines the equilibrium price, a number of qualitative results emerge. First of all,

$$\frac{dp}{d\gamma} = -\frac{(p - S)^2}{Z_{bt} h'(p)} < 0$$

indicating that tracking error considerations decrease the return on risk, as expected. Again in line with expectations,

$$\text{sign}\left[\frac{dx}{d\gamma}\right] = \text{sign}\left[p(2S-p)\frac{dp}{d\gamma}\right] = \text{negative}$$

i.e. MV types take less risk, the more TE types care about tracking error. Finally, the TE types may actually take less risk, the more intensely they pursue tracking error:

$$\text{sign}\left[\frac{dy}{d\gamma}\right] = \text{sign}\left[\left(1+\frac{\gamma}{Z_{bt}}\right)(2S-p)\frac{dp}{d\gamma} + p(p-S)\frac{1}{Z_{bt}}\right]$$

which is of ambiguous sign. Valid parameterizations exist that make this expression positive, and others exist that make it negative.

Result 6. *The equilibrium effect of tracking-error on quantities and price in any given period is as follows:*

$$\gamma\uparrow \Rightarrow p\downarrow, x\downarrow, \text{ and } y\uparrow \text{ or } \downarrow$$

3.2 *Comparing wealth accumulations in equilibrium*

This section examines wealth accumulations in equilibrium. First, an analytical result is presented showing that a benevolent planner attempting to maximize aggregate wealth will have the two agents specialize completely: one in the safe asset, the other in the risky asset. This, of course, minimizes the social costs of risk-taking and exploits the complementarity of the risky and safe activities. Note that the benevolent planner is not maximizing a weighted function of each agent's utility, but is maximizing aggregate wealth under the assumption (which holds in the initial period) that each type holds 50% of aggregate wealth.

The benevolent planner chooses x and y in the unit interval to maximize $E[p(x+y)+S(2-x-y)]$ which, after substitutions, is

$$(m-S)(x+y) - m\left(x^2y + xy^2\right) + 2S \tag{15}$$

To maximize this objective, either $(x, y)=(0, 1)$ or $(x, y)=(1, 0)$ must be chosen. These strategies ignore the risk aversion of one type and ignore the mean-loving preference of the other, and therefore conflict with the individual utility-maximizing decisions. What is noteworthy about tracking error preferences in this regard is that they promote specialization. Beyond the simple difference in strategy that arose directly out of the heterogeneity of the two types, the equilibrium price mechanism leads to a much more specialized pattern of risk-taking.[3]

[3]The constraint $R_{t+1} = e^{\varepsilon_{t+1}}(1-xy)$ captures the idea that the opportunity to profit from risk-taking diminishes when both TE and MV types pursue those risky opportunities. What is not taken into account by this constraint is the extent to which the number of TE types or number of MV types affects the returns process. This is an important limitation of the analysis in this paper. It remains to model this economy where these coordination costs depend on the number of or relative wealth of each type. The following alternative formulations of the constraint are possible candidates:

$$R_{t+1} = e^{\varepsilon_{t+1}}\left(c - s_x x - s_y y\right), \text{ or } \mathrm{E_t}\varepsilon_{t+1} = \mu_0\left(\mathrm{c} - \mathrm{s_x x} - \mathrm{s_y y}\right)$$

where s_x and s_y are the shares (at time t) of aggregate wealth held by MV and TE types respectively. In such a set-up, the conjecture that long-run equilibrium would feature positive numbers of each type, with a population ratio that is endogenously determined by taste-, innovation-process-, and tracking error parameters seems reasonable. The details have yet to be worked out, however, due, in part, to the difficulty of knowing how to handle non-uniqueness of equilibrium price.

To see this, a second set of simulations are presented for the equilibrium model. Unless noted otherwise the parameter values are centred at

$$\mu = 0.07\ \sigma = 0.10\ \beta = 37.31\ \gamma = 5\ \rho = 0\ A = 0,\ T = 100, \text{ and } S = 1.02$$

The value $\beta = 37.21$ is set so that, in isolation, MV types optimally choose to invest half their wealth in risk.

As illustrated in Figure 3, risk-taking is dramatically different in the equilibrium model. TE types put slightly less than half their wealth in risk, while MV types put only 8.5% of their wealth in risk. Even a slight weight on TE considerations is amplified through equilibrium channels so that TE types typically allocate five to six times as much of their wealth (in percentage terms) than MV types do. These figures are stable through time and robust across the specification of benchmarks. 6

Figure 3 shows the empirical distribution of the TE type's share of terminal aggregate wealth. TE types clearly end up with more wealth on average. But there are considerably more price paths in the equilibrium model for which TE types actually do worse than MV types. Figure 3 indicates that a number of monotonic relationships hold in equilibrium. In particular, increasing the risk premium, increasing volatility, increasing risk aversion, or increasing returns persistence all work to the advantage of TE types. These relationships (for volatility, risk aversion and persistence) are opposite of the non-equilibrium case.

3. Conclusions

This paper develops a simple model in which investors with heterogeneous preferences can be compared in terms of risk-taking and accumulated wealth. In particular, two types of investors are studied: traditional mean-variance investors, and a second class of investors who care about their portfolio returns relative to a benchmark level of performance, referred to in this paper as 'tracking error'.

Tracking error decision-makers try to make portfolio decisions that, in addition to yielding high return with low risk, are expected to beat a (personal, historical, or a competitor's) benchmark. Not only do tracking error decision-markers care about beating the benchmark, they also try to avoid lagging behind it. After calibrating risk aversion across the two types so that both have identical preferences when tracking error is zero, the paper demonstrates how tracking error decision-makers take more risk on average than mean-variance decision-makers do, and accumulate a greater share of aggregate wealth than mean-variance types do. Although this result is trivial for the isolation model when tracking error enters utility linearly, the non-linear case and the linear case in equilibrium are less obvious.

In isolation, non-linear tracking error preferences create a preference for positively skewed returns distributions, but can sometimes make decision-makers more risk averse. The net effect is ambiguous for any given period, because the time t preferences depend on last period's realized benchmark. For some values, the tracking error type will take fewer risks than the mean-variance type. But on average, the paper shows that tracking error leads to greater risk-taking and a greater share of wealth.

In equilibrium, although it is trivial to show the tracking error types take more risk, the net social effect of this is ambiguous, because there are social costs borne by all investors when aggregate risk-taking increases. The intensity of tracking error preference parameters γ has a clearly negative effect on the mean-variance types' demand for risk. But the tracking error types themselves may demand more or less risk when they care more about beating the benchmark.

After accounting for the crowding out effects by which risk-taking tracking error types lower the return on risk for everyone, the simulations in this paper find that average terminal aggregate wealth is increasing in the degree to which TE types care about tracking error. In other words, tracking error usually leads to net social benefits in the form of higher average levels of accumulated wealth, even though TE types' increased risk-taking imposes social costs.

This last result prompts us to conjecture that tracking error decision-makers are selected by an evolutionary mechanism where fitness is an increasing function of accumulated wealth. An individual whose parents were mean-variance investors but who, due to a mutation, cares about tracking error, winds up helping society in the individualistic pursuit of higher average returns. Assuming complementarity between risky and safe activities, simulations suggest that a pattern of specialization emerges which helps push aggregate returns closer to the social optimum.

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Appendix: Computing the Coefficients in $u_t^{\mathrm{TE}}[Z_{t+1}(x_{\mathrm{t}})] = a_0 + a_1 x_t + a_2 x_t^2 + a_3 x_t^3$

The goal is to compute $[a_0, a_1, a_2, a_3]$ in terms of exogenously given μ, σ^2, β, ρ, γ, S and pre-determined (at time t) $M \equiv E_t R_{t+1} = R_t^{\rho} e^{\mu + \frac{1}{2}\sigma^2}$, $V \equiv \mathrm{var}_t(R_{t+1}) = R_t^{2\rho} e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$, and Z_{bt}. Suppressing subscripts and superscripts to write the TE type's decision variable as x, the TE type's third-order tracking-preference utility function can be expressed as:

$$\begin{aligned} u(x) &= E_t Z_{t+1} - \frac{\beta}{2}\mathrm{var} Z_{t+1} + \gamma E_t (Z_{t+1} - Z_{\mathrm{b}t})^3 \\ &= (M - S)x + S - \frac{\beta}{2} V x^2 + \gamma E_t (Z_{t+1} - Z_{\mathrm{b}t})^3 \end{aligned} \tag{A1}$$

Using the earlier decomposition in equation (8), the cubic term in (A1) can be broken into pieces, expressed as polynomials in x by way of the following intermediate computations:

$$\begin{aligned} E_t(Z_{t+1} - E_t Z_{t+1})^3 &= E_t(R_{t+1} - M)^3 x^3 \\ &= (E_t R_{t+1}^3 - 3M E_t R_{t+1}^2 + 3M^3 - M^3)x^3 \\ &= (E_t R_{t+1}^3 - 3VM - M^3)x^3 \\ &= R_t^{3\rho} e^{3\mu + \frac{9}{2}\sigma^2} - 3VM - M^3)x^3 \\ 3E_t(Z_{t+1} - E_t Z_{t+1})^2 (Z_{t+1} - Z_{\mathrm{bt}}) &= 3V[x(M - S) + S - Z_{\mathrm{b}t}]x^2 \\ &= 3V(M - S)x^3 + 3V(S - Z_{\mathrm{b}t})x^2 \\ (E_t Z_{t+1} - Z_{\mathrm{bt}})^3 &= [(x(M - S) + (S - Z_{\mathrm{b}t})]^3 \\ &= (M - S)^3 x^3 + 3(M - S)^2 (S - Z_{\mathrm{b}t})x^2 \\ &\quad + 3(M - S)(S - Z_{\mathrm{b}t})^2 x + (S - Z_{\mathrm{b}t})^3 \end{aligned}$$

(The fourth equality above uses the result that $\log(Y^m) \sim \mathrm{N}(m\mu, m^2\sigma^2)$ for any log-normally distributed variable Y with parameters m and σ^2, together with the fact that $\log(R_{t+1}) \sim \mathrm{N}(\mu + \rho\log(R_t), \sigma^2)$, to derive $E_t(R_{t+1}^3) = e^{3[\mu + \rho \log(R_t)] + \frac{9}{2}\sigma^2}$.) The cubic term in (A1) can now be expressed as

$$E_t(Z_{t+1} - Z_{bt})^3 = [(R_t^{3\rho} e^{3\mu + \frac{9}{2}\sigma^2} - 3VM - M^3) + 3V(M - S) + (M - S)^3]x^3$$
$$+ [3V(S - Z_{bt}) + 3(M - S)^2(S - Z_{bt})]x^2 + 3(M - S)(S - Z_{bt})^2 x + (S - Z_{bt})^3$$

Returning to (A1) and summing up polynomial terms in x, the desired form is achieved, i.e. (A1) is re-written

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

with coefficients

$$a_3 = \gamma[R_t^{3\rho} e^{3\mu + \frac{9}{2}\sigma^2} - 3VM - M^3 + 3V(M - S) + (M - S)^3]$$

$$a_2 = -\frac{\beta}{2} V + 3\gamma[V + (M - S)^2](S - Z_{bt})$$

$$a_1 = (M - S)[1 + 3\gamma(S - Z_{bt})^2]$$

$$a_0 = S + (S - Z_{bt})^3$$

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