Media, Film and Communication

MFCO Working Paper Series

SPECIAL ISSUE: ENVIRONMENTS, SPACES AND TRANSFORMATIONS

Transforming a methodological landscape from deficit to growth in mathematics education research

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Abstract: Methodological legacies in mathematics education research have been challenged by contemporary perspectives and techniques for investigating students' mathematical knowledge. In this conceptual analysis, fluency was a central analytic concept used in a growth-oriented study of how students solved missing number problems. Specifically, students' mathematical knowledge was conceptualised as attempts to participate in mathematical activity rather than a lack of knowledge or demonstrations of faulty understandings fuelled by misconceptions. The methodology and findings from a recent study are used to illustrate how fluency can be used as an analytic concept to support a growth-oriented research approach.

Introduction

In this conceptual analysis, I examine the methodological landscape of mathematics education research and suggest a possible transformation. Methodology refers to the theory and methods used by researchers to conduct scholarly investigations. The methodology commonly used to investigate student knowledge of school mathematics is underpinned by a deficit orientation. Instead of adopting a deficit mode of inquiry, I present a research approach and findings from a study commensurate with contemporary learning theory and innovative teaching practice. The research approach views student mathematical knowledge from growth perspective and emphasises fluency. Fluency is used to recognise a student's demonstration of knowledge as an attempt to participate in mathematical activity rather than an amount of mathematical knowledge that is present or absent, and correct or incorrect. First, I outline how mathematics education research has been shaped by deficit oriented views of student knowledge and thinking. Second, I argue that a growth perspective affords opportunities for researchers to transform the methodological landscape in their fields of study. For mathematics education researchers in particular, the time is right for a methodological transformation. To illustrate how such a transformation can take place, I present examples from my doctoral study that demonstrate how a perspective based on growth and fluency can be put into practice.

Our Methodological Legacy

In mathematics education research, pathological shadows are cast across the methodological landscape by deficit language, methods, and theoretical orientations. Mathematics education research is an applied field of inquiry. Its parent disciplines are mathematics, psychology, and sociology. As researchers, we are wedged in-between the knowledge generated by the parent disciplines and the practice of mathematics instruction in schools. Our in-between role gives us the leverage to interpret and transform knowledge as it moves between practice and research contexts (Lerman 2000). As a community of researchers, however, we have still not fully capitalised on this opportunity (Skott, Van Zoest, & Gellert 2013). We have a history of using the research methods from the parent disciplines instead of developing our own approaches tailored to the unique environment of school mathematics.

In mathematics education research, error analysis is a well-established method for investigating students' mathematical knowledge and thinking. The history of error analysis in mathematics education research stretches back over the past nine decades (Radatz 1979, p. 163). An analysis of students' errors has also been promoted to teachers as a useful method for establishing a student's current level of mathematical knowledge or thinking (Clements 1980, p. 20). For example, students' written errors have been attributed to carelessness, a lack of motivation, faulty reading, a lack of reading comprehension, and inappropriate strategy or skill selection (Clements 1980, p. 15).

One form of deficit theorising can be tracked back to a hierarchical system for analysing errors made on written tasks published by Newman in 1977. Newman's procedure for error analysis identified specific and general ways that students could fail to successfully complete the five steps required for successful problem solving (Clements & Ellerton 1996, p. 4). The problem solving steps along with the types of errors attributed to students are summarised in Figure 1.

Step	**Problem Solving Description**	**Student Error Type**
1.	Read the problem	Lack of reading skills
2.	Comprehend what was read	Lack of comprehension about the mathematical ideas presented in the problem
3.	Transform ideas in the problem into an appropriate solution strategy	Lack of abstract analytical reasoning skills
4.	Execute the solution strategy	Lack of process skills
5.	Write a solution using appropriate notation	Lack of encoding skills

Figure 1.

Problem solving steps and types of errors attributed to students (Newman 1977 cited in Clements and Ellerton 1996, p. 4).

Note that the errors student made according to the Newman hierarchy are attributed due to a deficiency on the part of the student. Furthermore, carelessness and a lack of motivation were general failures that could occur during any step of the problem solving process. The language of errors, lacking, and carelessness, paints a bleak portrait of absence in students' mathematical knowledge in the context of problem solving.

Unfortunately, mathematics education researchers also use metaphors for student knowledge that exacerbate the effects of deficit methodology. While the notion of 'buggy' thinking has stimulated inquiry about students' mathematical knowledge for decades, the bug metaphor frames learners' thinking as potentially malfunctioning (e.g., Brown & Burton 1978; Hennessy 1994). When buggy thinking is detected, students are in need of 'de-bugging' just as faulty line of coding in a computer programme would need elimination and replacement. In contemporary studies of student knowledge, we continue the tradition of analysing students' error patterns and using negative psychological constructs (e.g., Linsell & Allan 2010; Linsell, Allan, & Anakin 2011; Linsell & Anakin, 2012). A phrase such as 'lack of knowledge' shows our focus on absence, and procedures that 'diagnose misconceptions' highlight our drive to identify and repair faulty mental representations. Our vocabulary and methods suggest that we have a penchant for pathology and it limits the ways we can theorise and investigate students' mathematical knowledge.

For example, when students and teachers talk about the mathematical concept of equality in arithmetic contexts, they use words such as 'equals', 'makes', and 'is the same as' which generally means that they are referring to the state of two quantities being the same. Students learn to use the symbol '=' to denote the concept of equality in mathematics lessons. Students may not be explicitly aware that the equals sign signifies a binary relationship between two statements that has three properties: reflexivity, symmetry, and transitivity (Jones 2009). The problem $7 + 3 = \square + 2$ is used to illustrate the three properties of equality. If a student recognises that $10 = 10$, $10 = 9 + 1$, $10 = 8 + 2$, and $10 = 7 + 3$, then that student is at least tacitly aware of the reflexive property of equality. If a student recognises that $7 + 3 = 10$ then $10 = 3 + 7$ then that student is at least tacitly aware of the symmetric property of equality. If a student recognises that $7 + 3 = 10$ and $10 = 8 + 2$, and then $7 + 3 = 8 + 2$; then that student is at least tacitly aware of the transitive property of equality.

Researchers have documented at least five types of conceptions that students hold of the equals sign. Four conceptions are associated with incorrect numerical responses whereas only one type has been associated with correct numerical responses. The greater number of incorrect conceptions than correct conceptions, and the label ‘common misconceptions’ suggests that researchers’ investigations have been largely deficit-oriented and error focused. One common misconception is when students interpret the equals sign as a signal to perform an action (Behr 1976), where the two numbers that precede the equals sign are added to calculate the missing number as 10 (i.e., 7 + 3 = 10 + 2). A second common misconception is when students interpret the equals sign as a prompt to execute a procedure (Kieran 1980), where all numbers in the problem are added and the answer is placed after the equals sign (i.e., 7 + 3 = 12 + 2). A third common misconception is when students interpret the equals sign as an operator-separator (Baroody & Ginsburg 1983,) where the equals sign may have several functions; as an addition symbol, as a place holder, and as a signifier that the solution to the problem follows (i.e., 7 + 3 = 10 + 2 = 12). A fourth common misconception is when students interpret the equals sign as a part of restricted notation (Seo & Ginsberg 2003), where the + 2 is disregarded and only the two addends and their sum that follows the equals sign are recognised as the components of the problem (i.e., 7 + 3 = 10).

The one conception of the equals sign associated with correct responses has been documented as a structural understanding (Brekke 2001; Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali 2006). However, the features of that conception have focused on the numerical solution to the problem (i.e., 7 + 3 = 8 + 2) rather than specifying the underlying mathematical knowledge that students are using to arrive at that solution. For the problem 7 + 3 = □ + 2 in particular, the important mathematical ideas involve students’ knowledge of number, operations, and equality. For example, a student could successfully solve the problem using their knowledge about quantitative sameness such that 7 + 3 = 10 and therefore 10 = 8 + 2 or 10 – 2 = 8. That same student, however, could also solve the problem by using alternate arithmetic knowledge that if 3 = (1 + 2) and (3 – 1) = 2 then 7 = (8 – 1) and (7 + 1) = 8. If the student gives the answer as 8, the general nature of the structural conception of the equals sign does not give specificity about whether the student has calculated a sum or used the strategy of compensation, and it has been suggested that the difference between those two conceptions is crucial for future learning in mathematics (Linsell 2010; Linsell, Allan & Anakin, 2011).

One result of the use of deficit theories in mathematics education is the unintended reinforcement of acquisitionist metaphor of learning. An acquisition metaphor underpinning mathematics instruction in schools becomes visible when teaching is dominated by the transmission of facts, from teacher to student, and when learning becomes the faithful reproduction of those facts, usually by a student rehearsing them in isolation (Sfard 1998). For example, a student giving the answer 12 to the problem $7 + 3 = \square + 2$ will be considered to have a misconception about the equals sign that needs eradicating with the insertion of a structural conception of the equals sign so that correct answer of 8 can be obtained.

When instruction is based on learning resources developed by mathematics education researchers that help teachers diagnose misconceptions, identify errors, or expose missing facts then their attention is focused on absence rather than on the capabilities students bring with them to learning mathematics at school. For example, the need to understand and diagnose students' misconceptions has found its way into the textbooks and the methods we used to teach our future mathematics teachers (e.g., Jorgensen & Dole 2011). However, a focus on errors and absence is not promoted by the New Zealand Curriculum (Ministry of Education 2007) or by the Numeracy Framework (Ministry of Education 2003). We are overdue, therefore, to question and challenge the pathological shadow that deficit methods cast on the methodological landscape of mathematics education research.

Cultivating a Growth Perspective

Recently, a provocative keynote address was given to members of the Australasian mathematics research community (Stillman 2013). The keynote address contained a clear message to re-examine our tacit use of established research tools when investigating the teaching and learning of school mathematics. The keynote address also implored us to look at using our research tools in alternate ways or even re-tooling them for different purposes. This call-to-action has been issued to the global mathematics education research community (Stillman 2014) and fundamentally, it echoes another rallying cry to set a growth-oriented agenda for teaching and learning of mathematics:

> The goal for mathematics instruction should be *meaningful engagement* with *powerful mathematics* for *all children* – resulting in children's development of the ability to engage in sense-making in and with mathematics, a deeper understanding of mathematical ideas, the ability to use mathematical ideas productively in solving problems, and a more positive view both of mathematics and of themselves as sense-makers in mathematics. (Schoenfeld 2012, p. 1, author's emphasis)

Schoenfeld's agenda for the mathematics education community is not new and it resonates with Lerman's analysis of the potential of mathematics education research community to innovate the teaching and learning of school mathematics issued a decade earlier (Lerman 2000). This growth-oriented agenda is also in alignment with the perspective demonstrated in Shulman's seminal work regarding the knowledge of teachers (Shulman 1986). In New Zealand, the Number Framework (Ministry of Education 2003) is an example of a growth-oriented form of knowledge for teachers based on a developmental model of students' thinking in arithmetic contexts.

The Number Framework was designed from observations of what children said and did as they worked mathematically. In 2003, the Number Framework was implemented in primary schools in New Zealand. In the decade that has elapsed since its first implementation, the use of the Number Framework has been followed up with studies that track its efficacy for teaching and learning mathematics in primary schools (e.g., Young-Loveridge 2010). However, developing instructional methods that are student-centred has been a challenge (e.g., Johnson, Thomas, & Ward 2010), in part due to the impact that a teachers' personal history as a mathematics student has on their practice as a mathematics teacher (Ball 1988). Not only do teachers carry the habits of their experiences as students learning school mathematics into their pedagogy as teachers, but they also bring emotional memories associated with learning school mathematics (Biddulph 1999). These experiences and memories contribute to teachers' positive and negative attitudes towards mathematics. Despite these challenges, the mathematics education community is poised to support growth and innovation in the teaching and learning of mathematics.

One way that mathematics education researchers can focus on growth and innovation involves a shift from acquisition to participation in the metaphors that we use for teaching and learning (Sfard 1998). Acquisition metaphors involve students passively receiving and faithfully reproducing knowledge transmitted to them by their teachers. When acquisition metaphors underpin researchers' studies then knowledge may be viewed as static and fixed. Participation metaphors involve interaction between students and teachers as they negotiate their understandings of knowledge together. Participation metaphors also re-position students as active agents in the instructional process. When participation metaphors underpin researchers' studies then knowledge may be viewed as dynamic and contextually situated.

A key feature of participation metaphors is that they may enable teachers to develop a growth perspective towards their students. With a growth perspective, teachers view the current capabilities of a learner as a starting point for developing further mathematics knowledge (Linsell & Anakin 2013 p. 445). Researchers in the discipline of psychology have recognised the damaging nature of negative psychological constructs and are actively promoting positive-orientations in their investigations (Seligman 1999). Likewise, mathematics education research tempered with a growth perspective can be used to investigate student knowledge at attempts to participate in mathematical activity rather than to expose deficits (Anakin 2013). Research goals can be transformed into investigations of fluency and appropriateness of knowledge communication rather than accounts of the volume and accuracy of knowledge reproduced (Anakin & Linsell 2014). An example follows of how a mathematics education research study was conducted with growth perspective and used the concept of fluency as an analytic tool.

Researching Student Knowledge as Fluency

In response to the error-focused nature of students' conceptions of the equals sign, my goal was to develop an approach that would frame a student's knowledge as increasing participation and fluency in mathematical activity rather than an amount of mathematical knowledge that was present or absent, and correct or incorrect. The analytical focal point of my study was the concept of fluency. Fluency represented the mathematical structure expressed in students' answers and explanations to the missing number problem, $7 + 3 = \square + 2$. Mathematical structure refers to the possibilities represented within and between the numbers, the operation of addition, and the relation of equality that are recognised and expressed by a student while solving the additive missing number problems used in this study (Mason, Stephens, & Watson 2009, p. 8). In relation to the missing number problem 7 + 3 = □ + 2, students may recognise possible number relations within the numbers such as 3 = 2 + 1, 3 – 1 = 2, 7 + 1 = 8 and 7 = 8 - 1. Student may recognise possible number relations between numbers in the problem such as 7 + 3 = 10 and 10 – 2 = 8. Student may recognise possible relations involving the operation of addition such as 7 + (1 + 2) = 10 and (7 + 1) + 2 = 10. And students may recognise possible relations involving equality such as 7 + 3 = 10 and 10 = 8 + 2, or (7 + 1) + (3 - 1) = 8 + 2.

Previously, the transition between arithmetic and algebra has been theorised as a didactic cut (Herscovics & Linchevski 1994). However, contemporary mathematics education researchers view algebra as generalised arithmetic (Carraher & Schliemann 2007) and have suggested that a thorough knowledge of/in? Arithmetic can provide a foundation for appreciating algebra (Mason, Stephens, & Watson 2009). Furthermore, innovative combinations of research techniques have been used to examine students' transition from arithmetic to algebra learning contexts (Caspi & Sfard 2012). Interpreting data from a student-oriented perspective is in alignment with the teaching philosophy enacted in contemporary mathematics classrooms in New Zealand (Ministry of Education 2003, 2007) and was a central feature in the design of this study. One aim of my study was to investigate the mathematical structure expressed by primary students as they sovled the problem, 7 + 3 = □ + 2.

My study design involved examining extant data collected by the Educational Assessment Research Unit at the University of Otago as part of the National Educational Monitoring Project (Crooks, Smith, & Flockton 2010). The National Educational Monitoring Project was an annual research and evaluation study conducted in New Zealand from 1996 to 2010. In 2009, over 4000 students in Year 4 (8-9 year olds) and Year 8 (12-13 year olds) took part in an assessment of their mathematical knowledge by completing a series of tasks that included the topics of number knowledge and algebra.

From the nationally representative sample, I randomly selected equal numbers Year 4 (N = 132) and Year 8 (N = 132) students that had completed the same missing number task. Students were asked to solve a set of additive missing number problems in a one-to-one assessment context with a teacher-assessor that was video recorded. Specifically, the

Year 4 and Year 8 students were given the problem, 7 + 3 = □ + 2, asked 'What is the missing number?' and then prompted with 'Explain why you say that'.

The data were analysed using multiple techniques (Drew & Heritage 1992; McNeill 1992; Roth 2001; Siegler & Crowley 1991). The video recordings were interpreted using micro-analytic techniques that involved analysing elements of students' verbal and non-verbal communication as they offered their responses to the question and prompt. To visualise the verbal and non-verbal ways that students expressed their responses, a mapping technique was used (Anakin 2013). The mapping techniques were developed from computational linguistics (Grune & Jacobs 2008) but applied in a mathematics language context (e.g., Caspi & Sfard 2012).

A general inductive approach (Thomas 2006) was used to examine the mathematical structure that each student expressed about the concept of equality and to make comparisons between the structures expressed by other students in the study. Importantly, the ways students expressed their answers and explanations were used to form the analytic categories rather than imposing a set of pre-determined criteria. Instead of categorising a student's response as correct or incorrect, the appropriateness of each student's response was characterised in terms of mathematical fluency. In order to meet the assumptions of the post-hoc statistical tests performed with the data, answers and explanations that occurred with a frequency of at least five in both Year 4 and Year 8 groups formed unique categories. Answers and explanations occurring with a frequency less than 5 were grouped into a category labelled Other Number and Ambiguous, respectively. Descriptive and inferential statistics were used to analyse the frequencies of students' common response patterns.

As expected, more Year 8 students than Year 4 students successfully solved the problem. Year 8 students offered 28% more appropriate answers and 28% more fluent explanations than Year 4 students. Accordingly, the frequencies of answer and explanation types are presented in decreasing order according to frequency in the Year 8 group. Because the emphasis of this study was on the fluency of students' responses, the results are reported in terms of what students were able to say and do rather than what they did not say and did not do.

When students in Year 4 and Year 8 were shown the problem 7 + 3 = □ + 2, asked 'What is the missing number?', they gave responses that were interpreted as 10, 8, 12, and Other Number answers. Students also gave responses that were interpreted as Ambivalent answers. The most frequent type of answer for the Year 8 group was the answer 10, instead of the appropriate answer of 8. The types of answers are defined below by representative examples of students' responses.

10. The verbal response 'ten' (S4022) was interpreted to express that the missing number could be 10. Fewer Year 8 students (n = 55) than Year 4 (n = 72) students gave 10 as their answer.

8. The verbal response 'eight' (S8036) was interpreted to express that the missing number could be 8. More Year 8 students (n = 53) than Year 4 (n = 16) students gave 8 as their answer.

12. The verbal response 'twelve' (S4004) was interpreted to express that the missing number could be 12. Fewer Year 8 students (n = 9) than Year 4 (n = 18) students gave 12 as their answer.

Ambivalent. A student's response was identified as an Ambivalent type of answer in two cases. In the first case, the verbal and non-verbal response was:

> I don't know… it's hard [right index finger touches the number 2] because [right index finger points to the number 2[the [right index finger touches the number 2] two's there I know what [right index finger placed on the number 3] that equals [right index finger lifted] but [right index finger placed on the number 2] do you have to add [right index finger travels to the number 7] that in [right index finger travel stops and points to the number 7] with [right index finger travels to and points at the number 3] them [right index finger travels to and placed on the number 2] or does it have to equal that [right index finger removed from the number 2 and shakes her head then raises then lowers shoulders] (S4050)

The response was interpreted to express that the student had no preference for a particular number. In the second case, the verbal response 'hm I think it's um ten…because it's seven plus three but it's also plus two wait make that twelve' (S4107) was interpreted to express that the student had changed the number during her response. Fewer Year 8 students (n = 8) than Year 4 (n = 11) students gave Ambivalent answers.

Other Number. The verbal responses ‘one’ (S8027), ‘two’ (S4048), ‘three’ (S4071), ‘four’ (S4030), ‘five’ (S8115), ‘six’ (S4014), ‘nine’ (S4019), and ‘thirteen’ (S4127) were interpreted to express that the missing number could be 1, 2, 3, 4, 5, 6, 9, and 13, respectively. Fewer Year 8 students (n = 7) than Year 4 (n = 15) students gave Other Number answers.

Because the students in this study were from a nationally representative sample, the data were also examined using inferential statistics. Inferential statistics allow predictions to be made about Year 4 and Year 8 students in New Zealand beyond the sample of students represented in this study. The chi-square (χ^2) statistic is a tool for examining the differences between the frequencies we observe and the frequencies we expect for each answer type (Smith, Gratz, & Bousquet 2009). The χ^2 test for independence was used to examine the relationship between the frequencies of answer types and year group of the students. The null hypothesis was that answer types were not related to year group. The alternative was that answer types were related to year group. The data met the assumptions of the χ^2 test for independence because each student’s answer was recorded as one answer type and year group only. Also, answer types were measured as frequencies and no expected frequency for any answer type was less than 5. According to the χ^2 test for independence, there was a significant statistical difference for answer types between the year groups, $\chi^2(4, N = 264) = 28.50, p < .01,$[1] therefore the null hypothesis was rejected. So from this sample, it was inferred that Year 8 students were more likely than Year 4 students to give 8 as an answer. Year 4 students were more likely than Year 8 students to give 10, 12, Ambivalent, or Other Number answers. Representative transcript extracts and the frequencies of Year 4 and Year 8 students’ answer types are presented in Table 1.

Table 1

Answer Types for 7 + 3 = □ + 2

Representativ	Answer Type	Year 8	Year 4

e Transcript Extract		n (%)	n (%)
ten (S4022)	10	55 (42%)	72 (55%)
eight (S8036)	8	53 (40%)	16 (12%)
twelve (S4004)	12	9 (7%)	18 (14%)
ten…wait…twelve (S4107)	Ambivalent	8 (6%)	11 (8%)
one (S8027)	Other Number	7 (5%)	15 (11%)
	Total	132 (100%)	132 (100%)

Note. *n* = number of students' responses. (%) = percent of students' responses. Words in italics represent students' verbal responses.

In general, when students in Year 4 and Year 8 were prompted with 'Explain why you say that', they gave responses that were interpreted as one of five explanation types. While the most fluent type of explanation was the most frequent in the Year 8 group, it was the fourth most frequent type of explanation in the Year 4 group.

Equals 10 Pair. The verbal response 'seven plus three is ten and eight plus two is ten' (S8001) was interpreted as an Equals 10 Pair explanation type. Equals 10 Pair explanations represented a comparison between two equality statements involving the number properties of 10 and the operation of addition. More Year 8 students (n = 57) than Year 4 (n = 19) students gave Equals 10 Pair explanations.

Equals 10. The verbal response 'seven plus three is ten' (S4036) was interpreted as an Equals 10 explanation type. Equals 10 explanations represented an equality statement involving the number properties of ten and the operation of addition. Fewer Year 8 students (n = 38) than Year 4 (n = 54) students gave Equals 10 explanations.

Equals 10 and 12. The verbal and non-verbal response 'seven plus three equals ten plus two equals [left hand raised and held above the problem then with left palm open, the left hand travels away and fingers are extended then left hand is removed] twelve' (S8007) was interpreted as an Equals 10 and 12 explanation type. Equals 10 and 12 explanations represented an equality sequence involving the number properties of 10 and 12, and the operation of addition. Fewer Year 8 students (n = 25) than Year 4 (n = 33) students gave Equals 10 and 12 explanations.

Ambiguous. A student's response was identified as an Ambiguous type of explanation if it was interpreted to be an Other or Partial type of explanation. The verbal response 'two plus one equals three' (S4016) was interpreted as an Other explanation type because it occurred with a frequency less than 5. The verbal response 'it equals it' (S4025) was interpreted as a Partial explanation type because it contained qualities that could be associated with more than one explanation type. Ambiguous explanations represented infrequent equality statements and partially expressed number properties. Fewer Year 8 students (n = 7) than Year 4 (n = 21) students gave Ambiguous explanations.

Equals 12. The verbal response 'seven plus three is ten and ten plus two is twelve' (S4024) was interpreted as an Equals 12 explanation type. Equals 12 explanations represented two equality statements involving the operation of addition and the number properties of 10 and 12, respectively, where there was no comparison made between the two equality statements. The same number of Year 8 students (n = 5) as Year 4 (n = 5) students gave Equals 12 explanations.

According to the χ^2 test for independence, there was a statistically significant difference found for explanation types between the year groups, $\chi^2(4, N = 264) = 29.89$, $p < .01$. So from this sample, it was inferred that Year 8 students were more likely than Year 4 students to give Equals 10 Pair explanations. Year 4 students were more likely than Year 8 students to give Equals 10, Equals 10 and 12, or Ambiguous explanations. Representative transcript extracts and the frequencies of the types of explanations for each year group are presented in Table 2.

Table 2

Frequencies of Explanation Types for 7 + 3 = □ + 2

Representative Transcript Extract	Explanation Type	Year 8 *n* (%)	Year 4 *n* (%)
seven plus three is ten and eight plus two is ten (S8001)	Equals 10 Pair	57 (43%)	19 (14%)
seven plus three is ten (S4036)	Equals 10	38 (29%)	54 (41%)
seven plus three equals ten plus two equals...twelve (S8007)	Equals 10 and 12	25 (19%)	33 (25%)
it equals it (S4025)	Ambiguous	7 (5%)	21 (16%)
seven plus three is ten and ten plus two is twelve (S4024)	Equals 12	5 (4%)	5 (4%)
	Total	132 (100%)	132 (100%)

Note. *n* = number of students' explanations. (%) = percent of students' explanations. Words in italics represent students' verbal responses.

When shown the problem 7 + 3 = □ + 2 and asked 'What is the missing number?', students' answers were categorised as five student-defined types rather dichotomously as correct or incorrect. The student-defined answer types were 8, 10, 12, Other Number, and Ambivalent. While the 8, 10, and 12 answer types have been previously documented and theorised in the mathematics education literature as a structural understanding (Brekke

2001; Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali 2006), as a signal to perform action or restricted notation (Behr 1976; Seo & Ginsberg 2003), and as a prompt to execute a procedure or as an operator-separator (Kieran 1980; Baroody & Ginsburg 1983), respectively, the ambivalent answer type has not. Because students may not be explicitly aware that the equals sign signifies a binary relationship between two statements that has three properties: reflexivity, symmetry, and transitivity (Jones 2009), ambivalent answers may represent a fragile state of students' knowledge as they recognise different combinations of number, operation, and equality possibilities within the problem.

When prompted with 'Explain why you say that', students' explanations were categorised as five student-defined types rather as one of the four previously defined misconceptions associated with incorrect answers or a general structural type that defined correct conceptions of the equals sign. The Equals 10, Equals 10 and 12, and Equals 12 types of explanations could be viewed as further evidence to confirm that students view the equals sign as a signal to perform action or restricted notation (Behr 1976; Seo & Ginsberg 2003), as an operator-separator (Baroody & Ginsburg 1983), and as a prompt to execute a procedure (Kieran 1980), respectively. However, misconceptions do not necessarily allow researchers to focus on what students know and can do. When those three types of explanations were considered from a standpoint that views them as attempts to participate in mathematical activity, then the emphasis shifted to the combinations of number, operation, and equality possibilities that students were able to recognise and express about the problem. For example, an Equals 12 type of explanation represented successful communication about two equality statements involving the operation of addition and the number properties of 10 and 12, respectively. A next learning step for students giving Equals 12 explanations could be to recognise and express the number properties of 10 only.

To expand the fluency of students giving Equals 12 explanations, students and teachers could also look into the language and mathematical ideas that are expressed in Equals 10 Pair explanations. The Equals 10 Pair type gives specificity to the general description of structural types (Brekke 2001; Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali 2006). In particular, the Equals 10 Pair type illustrates that even in Year 8, the way students recognise and work with the possible combinations of knowledge about numbers, operations, and equality differ in logic from how students might be taught to solve a missing number problem (Ministry of Education 2003). Students might be taught to work with the difference in the two expressions using a compensation strategy such as 3 = (1 + 2) and (3 – 1) = 2 then 7 = (8 – 1) and (7 + 1) = 8. However, Year 4 and Year 8 students whose explanations were categorised as Equals 10 Pair types, overall, were comparing the quantities of two equality statements such as 7 + 3 = 10 and 8 + 2 = 10.

Additionally, the ambiguous type of explanation has not been documented in the literature. The ambiguous type of explanation represents how a growth-orientated view of student communication can lead to acknowledgement of students' attempts to participate in mathematical activity rather than to categorise those explanation as errors, misconceptions, or a lack of knowledge. Instead of correcting, 'debugging', or transmitting knowledge to a student who does not successfully solves the problem, the concepts of fluency and ambiguity recognise the language and mathematical ideas present in answers and explanations as resources for learning.

The answers and explanations that students gave for the problem 7 + 3 = □ + 2 reflect how they recognise and express the mathematical structure of the problem. Students' explanation patterns differ in logic from what a mathematical approach would suggest. These findings are supported by other recent studies (i.e., Carraher & Schliemann 2007; Caspi & Sfard 2012) that view algebra as generalised arithmetic and challenge the view that arithmetic and algebra are separated by a didactic cut (Herscovics & Linchevski 1994).

The data in this study were interpreted from a student-oriented and socio-cultural perspective which is in alignment with the teaching philosophy enacted in contemporary mathematics classrooms in New Zealand (Ministry of Education 2003, 2007). Findings from this study can be used by researchers to enhance their study of teaching and learning of mathematics in general and of arithmetic as generalised algebra in particular. For example, the mathematical structure expressed by students about problem can be used by researchers to pinpoint their current fluency and next steps for learning. If a student explains their answer as 7 + 3 + 2 = 12 then that student recognises the number properties of 12 in the three numbers given in the problem, but has yet to recognise that the position of the equals sign in relation to those number changes requires different sets of number properties to be considered. Literature suggests that students will benefit from experiencing a variety of problem structures, especially those where the missing number occurs in positions other than following the equals sign (i.e., □ + 3 = 8 + 2, 7 + □ = 8 + 2, and 7 + 3 = 8 + □) (Mason, Stephens, & Watson 2009). The results from this study suggests that many New Zealand students in Year 4 and Year 8 are not recognising the unknown in different positions and increasing the range of missing number problems students experience may help to address this need. Also, for those students who gave the answer 8, their explanations were interpreted to express the concept of quantitative sameness, rather than the strategy of compensation. While quantitative sameness is an efficient problem solving strategy for arithmetic problems it becomes a barrier for students in algebraic contexts. In algebra it is the general relations between symbols rather than specified quantities that are emphasised. Therefore, when solving algebraic problems where strategies like compensation are useful for solving problems with unknowns. For example, $a + 3 = b + 2$ can be solved by recognising that compensation can be used to create equivalent statements such as $a + 3 – 3 = b + 2 – 3$ so that $a = b – 1$. Thus for any value of a, b will always be 1 less than a. By using a growth-oriented methodology and the concept of fluency, researchers examine how students develop their understanding of equality, missing number positions, and the strategy of compensation.

By researching student knowledge as fluency, I have addressed the repeated calls made to the mathematics education community to participate in research with a growth-oriented agenda (Lerman 2000; Schoenfeld 2012; Stillman 2013, 2014). The mathemataics education research community has recognised that student benefit as learners when they are positioned as active agents in the instructional process (Sfard 1998). In addition to the challenge of fostering positive attitude towards mathematics (Biddulph 1999), teachers have the hurdle of developing instructional practices that often differ from their own experiences as learners of school mathematics (Ball, 1988). Mathematics education researchers can also assist students and teachers in the enterprise of teaching and learning mathematics by acknowledging the contributions made by earlier student-centred research efforts (Brown & Burton 1978; Clements 1980; Clements & Ellerton 1996; Hennessy 1994; Radatz 1979) and by transforming those approaches from deficit-focused to growth-oriented (Anakin & Linsell 2014; Linsell & Anakin 2013). Instead of diagnosing students' errors, categorising misconceptions, fixing 'buggy thinking', and identifying the absence of knowledge, we can join researchers beyond our own discipline to investigate phenomena from a positive perspective (Seligman 1999). In my doctoral study, a growth-oriented theoretical framework allowed student knowledge to be investigated as fluency. A fluency conception of student knowledge was presented as one possible way that the methodological landscape of mathematics education research can be transformed from deficit-focused to growth-oriented.

Summary

In this conceptual analysis, the methodological landscape of mathematics education research was examined and a possible transformation using a practical example was suggested. The theory and methods commonly used to investigate student knowledge of school mathematics were shown to be underpinned by a deficit view of students' knowledge. An unintended consequence of a deficit methodological orientation was the reinforcement of transmission teaching and reproduction leaning with an emphasis on student errors, misconceptions, and 'buggy thinking'. To illustrate an alternate mode of inquiry, I presented an investigation from my doctoral study. The example of how I investigated students' conceptions of the equal sign used growth-oriented methodology to show how inquiry can be commensurate with socio-cultural learning theory and student-centred teaching practice. Student knowledge was investigated as attempts to participate in mathematical activity rather than an amount mathematical knowledge that was present or absent, and correct or incorrect. Fluency was introduced as an alternative analytic concept to the amount of knowledge that a student demonstrates. This growth-oriented, socio-cultural research approach challenges the theory and methods commonly used to investigate student knowledge of the equals sign. A growth perspective affords researchers opportunities to transform the methodological landscape in their fields of study, and in particular, the field of mathematics education research is ripe for such a transformation.

Biography

Megan Anakin is a PhD candidate in the field of mathematics education at the University of Otago College of Education. She is studying the relationship between student knowledge and achievement. Specifically, she is examining primary school students' conceptions of

equality in the context of additive arithmetic missing number problems. She has over twenty years' experience teaching in primary and secondary classrooms, in Canadian, the UK, and New Zealand. She has also been a teacher-researcher in a long-term action research project driven by three research questions. What makes the biggest difference to student learning? How do we know we are making a difference to student learning? Is our teaching practice aligned with the outcomes we want for our students?

Acknowledgements

The Educational Assessment Research Unit at the University of Otago collected the mathematics assessment data used in this study as part of the National Educational Monitoring Project. Access to the data, assessment materials, and assistance from the team has been gratefully appreciated.

Endnotes

1. 4 represents the degrees of freedom, N = 264 represents the sample of 264 students, 28.50 represents the calculated value of χ^2, and the $p < .01$ represents the probability of making a Type I error or rejecting the null hypothesis when it is true.

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