

A theoretical foundation for the Nelson and Siegel class of yield curve models

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Abstract

Yield curve models within the popular Nelson and Siegel (1987, hereafter NS) class are shown to arise from a formal Taylor approximation to the generic Gaussian affine term structure model outlined in Dai and Singleton (2002). That theoretical foundation provides an assurance that NS models correspond to a well-accepted framework for yield curve modeling. It further suggests that any yield curve from the GATSM class can be represented parsimoniously by a two-factor arbitrage-free NS model. Such a model is derived and applied to provide evidence for changes in United States yield curve dynamics over the period from 1971 to 2010.

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This article establishes a theoretical foundation for the popular Nelson and Siegel (1987, hereafter NS) class of yield curve models via the generic Gaussian affine term structure model (hereafter GATSM) outlined in Dai and Singleton (2002). In particular, the article explicitly shows how the Level, Slope, and Curvature components that are common to all models of the NS class arise as “optimal approximations”, in a sense defined further below, to the dynamic component of the generic GATSM.

The primary motivation for establishing this result is to tackle “the elephant in the room” issue that has been conveniently overlooked with the extensive application of NS models since their inception.¹ That is, the Level, Slope, and Curvature components common to all models of the NS class were only justified heuristically when first proposed. That heuristic basis is highlighted with a selection of quotations from the introduction and conclusion of the original NS article:

“The purpose of this paper is to introduce a simple, parsimonious model that is flexible enough to represent the range of shapes generally associated with yield curves: monotonic, humped, and S shaped.” “A class of functions that readily generates the typical yield curve shapes is that associated with solutions to differential or difference equations. The expectations theory of the term structure provides heuristic motivation for investigating this class since, if spot rates are generated by a differential equation, then forward rates, being forecasts, will be the solution to the equations.” “A more parsimonious model that can generate the same range of shapes is given by the solution equation for the case of equal roots.” “Our objective in this paper has been to propose a class of models, motivated by but not dependent on the expectations theory of the term structure, that offers a parsimonious representation of the shapes traditionally associated with yield curves.”

Even the recent introduction of arbitrage-free NS models (e.g. Sharf and Filipović 2004, Krippner 2006a, Christensen, Diebold and Rudebusch 2009, 2010), while at least imposing theoretical self-consistency by explicitly accounting for assumed Gaussian yield curve dynamics, still take the NS components as given.² Justification, if supplied, appeals to the practical benefits of NS models, such as their ease of estimation, close fit to the yield curve data, intuitive estimated components, and successful past applications.³ A complementary and arguably more compelling justification for NS models would be a foundation within a well-accepted set of principles and assumptions for modeling the yield curve and its dynamics.

To this end, section 1 specifies the generic GATSM from Dai and Singleton (2002) and then derives the associated forward rate curve. Section 2 shows how the original NS forward rate curve arises from the dynamic component of the generic GATSM forward rate curve using an “optimal approximation”; specifically a low-order Taylor expansion around central measures of the eigenvalues associated with the generic GATSM. The NS Level component is shown to correspond to the persistent (i.e. slowly mean-reverting, or near-zero eigenvalue) components of the generic GATSM, and the NS Slope and Curvature components are shown to correspond to the non-persistent (i.e. non-zero eigenvalue) components of the generic GATSM. In light of this example, section 3 discusses how most models within the NS class, with one notable exception being the Svensson (1995)/NS model, can be classified as various optimal approximations to

the generic GATSM. That classification system therefore provides a useful guide to selecting an appropriate NS model for a given application.

A corollary from the theoretical foundation and classification system for the NS class of models is the evident absence of a generic two-factor arbitrage-free NS model. Such a model is likely to prove useful as a standard tool for yield curve modeling and analysis in economics and finance due to its representation of the generic GATSM with just two factors. Therefore, regardless of the number of factors in any GATSM, their nature (e.g. economic, financial, etc.) and the potential complexity of their interactions, the Level component of the NS model will represent the persistent components of the GATSM, and the Slope component will represent the non-persistent components. Section 4 derives the arbitrage-free two-factor NS model for the cases of time-invariant and time-varying term premia. As an empirical illustration of the models, section 5 applies them to investigate changes in United States yield dynamics. Section 6 concludes.

1 The generic Gaussian affine term structure model

The generic GATSM specified in this section parallels the standard multifactor Gaussian dynamic term structure model as outlined in appendix A of Dai and Singleton (2002). It is the fully Gaussian subset of the affine framework outlined in Duffie and Kan (1996) with the essentially affine specification of market prices of risk from Duffie (2002). In the notation of Dai and Singleton (2000) the specification is $A_0(N)$.

Three points of context for this article are worth noting up front. First, while the state variables are completely generic, and so could represent points on the yield curve as in Duffie and Kan (1996), it is more convenient for the subsequent discussion in section 4 to consider them as (potentially unobserved) economic and financial factors within the underlying economy. This follows the Duffie and Kan (1996) p. 321 interpretation that the state variables in an affine model can always, under standard assumptions, be related back to economic factors (e.g. preferences, technology, consumption, inflation, etc.) within a general equilibrium model. For example, Berardi and Esposito (1999) provides a generic basis for multifactor GATSMs based on an economy of the Cox, Ingersoll and Ross (1985*a*) type. Regarding financial factors (e.g. default risk, liquidity risk, repurchase effects, etc.), Duffie and Singleton (1999) shows how they can be incorporated into the generic GATSM framework,⁴ and Singleton (2006) chapter 14 contains an extensive summary of that literature.

Second, to make the exposition more transparent from the perspective of the original NS model, this article derives and works with the forward rate curve associated with the generic GATSM. The affine term structure literature more commonly uses bond prices and/or interest rate curves, but all are within an elementary transformation of each other and are equivalent perspectives for representing the yield curve.

Third, being fully Gaussian, the results for relating the generic GATSM to NS models do not extend to term structure models with full Cox, Ingersoll and Ross (1985*b*)/square-root dynamics. Appendix A illustrates this by example, and briefly discusses the practical implications. In short, NS models inherit the same theoretical shortcomings of GATSMs (i.e. positive probabilities of negative interest rates and constant volatilities), and that perspective should be considered when deciding if it is appropriate to apply an NS model for the task at hand. That said, the assumption

of Gaussian dynamics is standard in economics and macrofinance (whether explicitly, or implicitly via the application of Gaussian-based econometrics),⁵ and this article presupposes from this point onward that the user has already assumed a Gaussian data-generating process for the yield curve.

Define the instantaneous short rate at time t as $r(t) = \xi_0 + \xi_1' X(t)$, where ξ_0 is a constant, $X(t)$ is an $N \times 1$ vector of state variables, and ξ_1 is a constant $N \times 1$ vector. Under the physical P measure, the state variables follow the process $dX(t) = K_P [\theta_P - X(t)] dt + \Sigma dW_P(t)$, where K_P is a constant $N \times N$ mean-reversion matrix, θ_P is a constant steady-state $N \times 1$ vector for $X(t)$, Σ is a constant $N \times N$ volatility matrix, and $W_P(t)$ is an $N \times 1$ vector of independent Brownian motions. Define the market prices of risk as $\Pi(t) = \Sigma^{-1} [\pi_0 + \pi_1 X(t)]$, where π_0 is a constant $N \times 1$ vector and π_1 is a constant $N \times N$ matrix. Under the risk-neutral Q measure, the state variables follow the process $dX(t) = K_Q [\theta_Q - X(t)] dt + \Sigma dW_Q(t)$, where $dW_Q(t) = dW_P(t) + \Pi(t) dt$, $K_Q = K_P + \pi_1$, and $\theta_Q = (K_P + \pi_1)^{-1} (K_P \theta_P - \Sigma \pi_0)$. Zero-coupon bond prices under measure Q for the generic GATSM are $P(t, T) = \exp [A^*(\tau) + B(\tau)' X(t)]$, where T is the time of maturity, $B(\tau) = [\exp(-K_Q' \tau) - I] (K_Q')^{-1} \xi_1$, τ is the time to maturity $\tau = T - t$ ($T \geq t$, $\tau \geq 0$), and I is the $N \times N$ identity matrix. The full expression for $A^*(\tau)$ is provided in Dai and Singleton (2002), but the present article requires only the summary results that $A^*(\tau)$ is required for the system to be arbitrage free, and it can be expressed in the functional form $-\xi_0 \tau + A(\tau)$.

From Heath, Jarrow and Morton (1992, hereafter HJM), instantaneous forward rates are defined as $f(t, T) = -\partial \log P(t, T) / \partial T$. Therefore, under measure Q , the generic GATSM forward rate curve is:

$$f(t, T) = \xi_0 + [\exp(-K_Q' \tau) \xi_1]' X(t) - \frac{\partial}{\partial \tau} A(\tau) \quad (1)$$

Now express K_Q' in eigensystem form; i.e. $K_Q' = Z \Psi Z^{-1}$, where Z is the $N \times N$ non-singular matrix of eigenvectors each normalized to 1, and Ψ is the $N \times N$ diagonal matrix containing the N eigenvalues $(\lambda_1, \dots, \lambda_n, \dots, \lambda_N)$. The latter are assumed to be unique and positive, which follows the standard assumption in Duffie and Kan (1996) and Dai and Singleton (2002). Hence, $\exp(-K_Q' \tau) = \exp(-Z \Psi Z^{-1} \tau) = Z \exp(-\Psi \tau) Z^{-1} = Z \Lambda Z^{-1}$, where $\Lambda = \text{diag}[\exp(\lambda_1 \tau), \dots, \exp(\lambda_n \tau), \dots, \exp(\lambda_N \tau)]$, an $N \times N$ diagonal matrix. The forward rates in equation 1 are then $f(t, T) = \xi_0 + \{Z \Lambda Z^{-1} \xi_1\}' X(t) - \frac{\partial}{\partial \tau} A(\tau)$. This can be expressed equivalently as:

$$f(t, T) = \xi_0 + \sum_{n=1}^{n_0} q_n(t) \exp(-\lambda_n \tau) + \sum_{n=n_0+1}^N q_n(t) \exp(-\lambda_n \tau) - \frac{\partial}{\partial \tau} A(\tau) \quad (2)$$

where the coefficients $q_n(t)$ associated with each unique $\exp(-\lambda_n \tau)$ represent the collection of coefficients of the $\exp(-\lambda_n \tau)$ terms that arise from the full matrix multiplication of $\{Z \Lambda Z^{-1} \xi_1\}' X(t)$.

For use in the example of the following section (but without loss of generality) it is assumed that the $q_n(t) \exp(-\lambda_n \tau)$ components have been re-ordered from the smallest to the largest eigenvalue, and then divided into two groups. The first group contains the components with eigenvalues λ_1 to λ_{n_0} that are close to zero (i.e. the persistent components, given they will have a slow exponential decay by time to maturity τ) and

the second group contains the eigenvalues λ_{n_0+1} to λ_N that are not close to zero (i.e. non-persistent components).

Also for use in the following section, the first three components of equation 2 are collectively denoted the dynamic component of the generic GATSM. This terminology reflects that the time series properties of the generic GATSM forward rate curve are completely contained in the $q_n(t) \exp(-\lambda_n \tau)$ components. That is, the coefficients $q_n(t)$ are linear combinations of the original state variables $X(t)$, and the functions $\exp(-\lambda_n \tau)$ determine the expected (as at time t) evolution of forward rates from time t to time $t + \tau$. As time evolves, the stochastic term $\Sigma dW_P(t)$ for the generic GATSM imparts innovations to the state variables $X(t)$, which are reflected as innovations to the coefficients $q_n(t)$. Conversely, the non-dynamic component $\frac{\partial}{\partial \tau} A(\tau)$ is a time-invariant function of time to maturity.

2 The generic GATSM to the original NS model

A formal Taylor approximation may be used to reproduce the original NS model from the exact expression of the dynamic component of the generic GATSM forward rate curve in equation 2. The treatment of the time-invariant component $\frac{\partial}{\partial \tau} A(\tau)$ is discussed further below.

For the first group of eigenvalues where $\lambda_n \simeq 0$, the first term of the Taylor expansion is $\exp(-\lambda_n \tau) \simeq 1$. For the second group of eigenvalues where $\lambda_n \gg 0$, express them relative to $\phi = \text{mean}(\lambda_{n_0+1}, \dots, \lambda_N)$,⁶ so that $\lambda_n = \phi(1 - \delta_n)$ and $\exp(-\lambda_n \tau) = \exp(-\phi \tau) \exp(\delta_n \phi \tau)$. From the latter expression, taking two terms of the Taylor expansion around δ_n gives $\exp(-\phi \tau) (1 + \delta_n \phi \tau)$, so $q_n(t) \exp(-\lambda_n \tau) \simeq q_n(t) \exp(-\phi \tau) + q_n(t) \delta_n \phi \tau \exp(-\phi \tau)$. Substituting these results into equation 2 gives:

$$f(t, T) \simeq \xi_0 + \sum_{n=1}^{n_0} q_n(t) + \left[\sum_{n=n_0+1}^N q_n(t) \right] \exp(-\phi \tau) + \left[\sum_{n=n_0+1}^N q_n(t) \delta_n \right] \phi \tau \exp(-\phi \tau) \quad (3)$$

This expression is precisely the functional form of the original NS model of the forward rate curve, i.e.:

$$f(t, T) \simeq f_{\text{NS}}(t, \tau) = L(t) + S(t) \exp(-\phi \tau) + C(t) \phi \tau \exp(-\phi \tau) \quad (4)$$

where $f_{\text{NS}}(t, \tau)$ adopts the typical time and time to maturity notation for NS models, 1, $\exp(-\phi \tau)$, and $\phi \tau \exp(-\phi \tau)$ are the forward rate factor loadings for the original NS model, and $L(t) = \xi_0 + \sum_{n=1}^{n_0} q_n(t)$, $S(t) = \sum_{n=n_0+1}^N q_n(t)$, and $C(t) = \sum_{n=n_0+1}^N q_n(t) \delta_n$ are the coefficients for the original NS model. The standard transformation $R_{\text{NS}}(t, \tau) = \frac{1}{\tau} \int_0^\tau f_{\text{NS}}(t, \tau) d\tau$ then produces the original NS model for the interest rate curve.⁷

The exposition above shows explicitly how the Level component of the original NS model approximates the constant plus the persistent dynamic components of the generic GATSM, and how the NS Slope and Curvature component approximate the non-persistent dynamic components. Moreover, the approximations are “optimal” in the sense that each additional NS component corresponds precisely to each successive term of the Taylor expansion around the central eigenvalues for the dynamic component

of the generic GATSM term structure. Conversely, other functions (e.g. see James and Webber (2000) chapter 15) cannot provide an “optimal approximation” in the sense already noted. For example, a third-order polynomial function would not be a precise third-order approximation (or even a precise zeroth-order approximation) of the non-persistent GATSM components.

The original NS model as derived above can be made arbitrage free (if required; see the discussion at the end of the following section) by adding appropriate terms to account for the effects that the market prices of risk and volatilities of the NS factor loadings have on the forward rate curve. For example, Christensen, Diebold and Rudebusch (2010) directly derives the arbitrage-free (hereafter AF) terms for $R_{NS}(t, T)$ via a particular three-factor GATSM designed to reproduce the original NS factor loadings, while Krippner (2006) calculates the AF terms via the HJM framework. Note that calculating the appropriate AF terms from a base NS model, by whatever means, guarantees the resulting model will be AF with respect to the NS components that are themselves an optimal approximation of the dynamic component of the GATSM. Conversely, a direct Taylor approximation of the generic GATSM including its AF terms would not necessarily guarantee a resulting AF model.

3 A GATSM perspective for classifying, selecting, and applying NS models

By following the example in the previous section, it is possible to classify most NS models as a particular Taylor approximation of the generic GATSM. The key aspects are: (1) the number of groups of non-zero eigenvalues assumed for the non-persistent dynamic components of the generic GATSM, which determines how many mean eigenvalue parameters are required; (2) the order of approximation chosen around each mean eigenvalue, which determines the number of components from the Taylor expansion associated with each mean eigenvalue; and (3) whether the AF term is included, which determines if the NS model is AF with respect to its factor loadings and the assumed specification for the market prices of risk.

From the perspective of these three aspects, table 1 summarizes the NS models already proposed in the literature. Permutations of the three aspects above can obviously generate an infinite variety of alternative NS models, but table 1 adds just three variants to cover evident absences in the range of proposed NS models to date, and then two further variants with six components for illustration.

[Table 1 around here]

As an example of interpreting table 1, the Diebold, Li and Yue (2008)/NS model with the forward rate form $f(t) = L(t) + S(t) \exp(-\phi\tau)$ is the most parsimonious representation of the generic GATSM.⁸ It represents both the persistent and non-persistent components of the generic GATSM with a single term from the Taylor expansion and omits any AF adjustments. Interestingly, the two-factor HJM model (HJM pp. 91-92) turns out to be an AF version of the Diebold et al. (2008)/NS model, by coincidence of the assumed volatility functions for the forward rate curve being a constant and an exponential decay by time to maturity. However, the HJM model assumes uncorrelated innovations; Variant 1 derived in section 4 effectively extends the HJM model to allow for correlation (and time-varying market prices of risk).

At the other extreme, the Christensen, Diebold and Rudebusch (2009)/NS model is the most comprehensive model within the NS class to date. It has the forward rate form $f(t, \tau) = L(t) + S_1(t) \exp(-\phi_1 \tau) + C_1(t) \phi_1 \tau \exp(-\phi_1 \tau) + S_2(t) \exp(-\phi_2 \tau) + C_2(t) \phi_2 \tau \exp(-\phi_2 \tau) + AF(\tau)$, where $AF(\tau)$ abbreviates the AF term from Christensen et al. (2009) in its forward rate form. In this model, the non-persistent components are represented with two groupings of non-zero eigenvalues (i.e. $\phi_1 = \text{mean}(\lambda_{n_0+1}, \dots, \lambda_{n_1}) < \phi_2 = \text{mean}(\lambda_{n_1+1}, \dots, \lambda_N)$ each with two terms, and the appropriate AF terms is included to ensure the model is AF with respect to the five factor loadings. Variant 4 is a potential extension of the Christensen et al. (2009)/NS model that represents the persistent GATSM components with two terms of the Taylor expansion, thereby replacing $L(t)$ with $L_1(t) + L_2(t) \tau$, where $L_2(t) = -\sum_{n=1}^{n_0} q_n(t) \lambda_n$.⁹ Variant 5 is a six-component NS model that adds the third terms of the Taylor expansion for both the persistent and non-persistent components, i.e. $L_3(t) \frac{1}{2} \tau^2$ with $L_3(t) = \sum_{n=1}^{n_0} q_n(t) \lambda_n^2$, and $C^*(t) \frac{1}{2} (\phi \tau)^2 \exp(-\phi \tau)$ with $C^*(t) = \sum_{n=n_0+1}^N q_n(t) \delta_n^2$.

Note that all of the variant NS models (and the two-component models) are “balanced” approximations of the generic GATSM, in the sense that the number of Taylor terms is the same for the persistent and non-persistent components. That property has some theoretical appeal, because it guarantees an approximation of any GATSM to a given order. However, the practical relevance of imposing the property is an open question, as is the feasibility and usefulness of estimating the additional components in the variant models.

The classification system suggests a systematic approach to selecting and applying an appropriate NS model for the application at hand rather than arbitrarily adding flexibility to seek a better fit to the yield curve data. Of particular note in this regard are the NS models of Bliss (1997), Svensson (1995), and Sharef and Filipović (2004): the addition of just the second Curvature term $C_2(t)$ cannot represent a first-order Taylor approximation to the component of the generic GATSM associated with the second group of non-zero eigenvalues. Hence, those models should be avoided if an explicit correspondence to the GATSM class is desired. Another aspect is more subtle: to maintain an explicit correspondence with the generic GATSM, NS models should be applied with a constant parameter ϕ (or parameters ϕ_1, ϕ_2 , etc.) because that corresponds to the constant parameters assumed in the generic GATSM.¹⁰

Once the appropriate NS model has been chosen, consideration needs to be given to whether the model is made AF by adding the AF term with respect to the NS components (as in Christensen et al. (2009, 2010), or directly via the HJM framework as in the example of section 4). Ideally, the AF term should be included in empirical applications to maintain theoretical consistency between the cross-sectional and time-series properties of the given NS model,¹¹ and the correspondence back to the GATSM. Explicit estimates of the market prices of risk and the volatilities associated with the AF property may also provide useful information to the user, and are certainly essential when pricing instruments that are heavily influenced by interest rate volatility, such as options on fixed interest securities.

That said, non-AF NS models are perfectly valid when applied in a time-series context, such as forecasting the yield curve, establishing relationships with macroeconomic time-series data, or generating zero-coupon interest rate data to be used subsequently in a time series context. That result follows from Joslin, Singleton and Zhu (2010), which shows that any vector autoregressive factor model of the yield curve (thereby

including non-AF NS models) is valid in a time-series context, and AF restrictions do not add any advantage to the time series and forecast properties.¹²

4 The two-factor arbitrage-free NS model

This section derives an AF NS model with just two factors, i.e. the Level and Slope components. This particular model is motivated by the theoretical foundation and classification discussed earlier, because it is evidently absent from the range of NS models already proposed, and because it is also the most parsimonious self-consistent representation of the generic GATSM. Therefore, for any GATSM model, regardless of the number of factors, their nature (e.g. economic, financial, etc.) and the potential complexity of factor interactions within the mean-reversion and innovation matrices, the Level component of the NS model will represent the persistent components of the GATSM, and the Slope component will represent the non-persistent components. The AF term allowing for correlated innovations to the Level and Slope components and a specification for the market prices of risk will enforce self-consistency between the cross-sectional and time-series properties of the model.

Section 4.1 derives the model with constant market prices of risk (hence time-invariant term premia), and is denoted the AF/NS(2) model. Section 4.2 extends the AF/NS(2) model to include time-varying market prices of risk (hence time-varying term premia) with the essentially-affine specification, and is denoted the EA/AF/NS(2) model.

4.1 The AF/NS(2) model

4.1.1 Model derivation

The NS model with two factors is $\beta_1(t) + \beta_2(t) \cdot \exp(-\phi\tau)$. The HJM framework with the modification from Tchuindjo (2008) allows the factors to have innovations $\Omega [dW_1(t), \exp(-\phi\tau) \cdot dW_2(t)]$ with a non-zero correlation, i.e.

$$\Omega = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad (5)$$

where σ_1 and σ_2 are the factor volatilities (annualized standard deviations of the factor innovations), ρ is the correlation of factor innovations, and $dW_1(t)$ and $dW_2(t)$ are independent Wiener increments.

The expression for the AF/NS(2) forward rate curve in the Tchuindjo (2008)/HJM framework is therefore:

$$\begin{aligned} f(t, \tau) = & [1, \exp(-\phi\tau)] \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} \\ & + \int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi s)] \begin{bmatrix} \gamma_{0,1} \\ \gamma_{0,2} \end{bmatrix} ds \\ & - \int_0^\tau \left\{ [\sigma_1, \sigma_2 \exp(-\phi s)] \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \left(\int_s^\tau \begin{bmatrix} \sigma_1 \\ \sigma_2 \exp(-\phi u) \end{bmatrix} du \right) ds \right\} \quad (6) \end{aligned}$$

where the first line is $\mathbb{E}_t[r(t + \tau)]$, the expected value, conditional upon information available at time t , of the instantaneous short rate at time $t + \tau$; the second line is the term premium function involving the factor volatilities and the constant market prices of risk $\gamma_{0,1}$ and $\gamma_{0,2}$; the third line is the volatility effect involving the factor volatilities and their correlation ρ ; and u and s are dummy integration variables. Note that $\gamma_{0,1}$ and $\gamma_{0,2}$ are defined in this article as positive quantities, so they (intuitively) add positive spreads to $\mathbb{E}_t[r(t + \tau)]$ and therefore the observed yield curve.

The solution to the forward rate expression in equation 6 is:¹³

$$\begin{aligned}
f(t, \tau) &= \beta_1(t) + \beta_2(t) \cdot \exp(-\phi\tau) \\
&\quad + \sigma_1\gamma_{0,1} \cdot \tau + \sigma_2\gamma_{0,2} \cdot F(\phi, \tau) \\
&\quad - \sigma_1^2 \cdot \frac{1}{2}\tau^2 - \sigma_2^2 \cdot \frac{1}{2}[F(\phi, \tau)]^2 \\
&\quad - \rho\sigma_1\sigma_2 \cdot \tau F(\phi, \tau)
\end{aligned} \tag{7}$$

where $F(\phi, \tau) = \frac{1}{\phi}[1 - \exp(-\phi\tau)]$. The expression for $f(t, \tau)$ may be more conveniently expressed as $f(t, \tau) = a(\tau) + b(\tau)\beta(t)$, where $a(\tau) = \sigma_1\gamma_{0,1} \cdot \tau + \sigma_2\gamma_{0,2} \cdot F(\phi, \tau) - \sigma_1^2 \cdot \frac{1}{2}\tau^2 - \sigma_2^2 \cdot \frac{1}{2}[F(\phi, \tau)]^2 - \rho\sigma_1\sigma_2 \cdot \tau F(\phi, \tau)$, $b(\tau) = [1, \exp(-\phi\tau)]$, and $\beta(t) = [\beta_1(t), \beta_2(t)]'$.

Interest rates, rather than forward rates, are the observables that define the yield curve in practice. Hence, the standard relationship $R(t, \tau) = \frac{1}{\tau} \int_0^\tau f(t, \tau) d\tau$ is used to derive the interest rate function as $R(t, \tau) = \bar{a}(\tau) + \bar{b}(\tau)\beta(t)$, where $\bar{a}(\tau) = \sigma_1\gamma_{0,1} \cdot \frac{1}{2}\tau + \sigma_2\gamma_{0,2} \cdot \frac{1}{\phi} [1 - \frac{1}{\tau}F(\phi, \tau)] - \sigma_1^2 \cdot \frac{1}{6}\tau^2 - \sigma_2^2 \cdot \frac{1}{2\phi^2} (1 - \frac{1}{\tau}F(\phi, \tau) - \frac{1}{2\tau}\phi[F(\phi, \tau)]^2) - \rho\sigma_1\sigma_2 \cdot \frac{1}{\phi^2} (1 - \frac{1}{\tau}F(\phi, \tau) + \frac{1}{2}\phi\tau - \phi F(\phi, \tau))$ and $\bar{b}(\tau) = [1, \frac{1}{\tau}F(\phi, \tau)]$.

The interest rate as a function of time to maturity τ will be composed of the average of the expected path of the short rate up to τ (a time-varying quantity), and a time-invariant term premium function:

$$\text{TP}(\tau) = \sigma_1\gamma_{0,1} \cdot \frac{1}{2}\tau + \sigma_2\gamma_{0,2} \cdot \frac{1}{\phi} \left[1 - \frac{1}{\tau}F(\phi, \tau) \right] \tag{8}$$

4.1.2 State space representation

An observation of zero-coupon continuously compounding yield curve data at time t may be represented as:

$$R(t) = \bar{A} + \bar{B}\beta(t) + v(t) \tag{9}$$

where $R(t)$ is the $K \times 1$ vector of yield curve data (with $K = 10$ or 12 in the empirical application below), \bar{A} is the $K \times 1$ vector $[\bar{a}(\tau_1), \dots, \bar{a}(\tau_k), \dots, \bar{a}(\tau_K)]'$, and \bar{B} is the $K \times 2$ matrix $[\bar{b}(\tau_1), \dots, \bar{b}(\tau_k), \dots, \bar{b}(\tau_K)]$, and $\tau_1, \dots, \tau_k, \dots, \tau_K$ are the times to maturity of the yield curve data.

The evolution of $\beta(t)$ over a finite time step Δt may be derived directly as:

$$\beta(t + \Delta t) = \Phi(\phi, \Delta t)\beta(t) + \varepsilon(t + \Delta t) \tag{10}$$

where $\Phi(\phi, \Delta t) = \text{diag}[1, \exp(-\phi\Delta t)]$, a 2×2 diagonal matrix.

Equations 9 and 10 are a measurement and state equation that may be used in the Kalman filter to estimate the model. Regarding the additional elements required for

the Kalman filter, the covariance matrix for the state equation may be evaluated as:

$$Q = \int_0^{\Delta t} \Phi(\phi, s) \Omega [\Phi(\phi, s)]' ds \quad (11)$$

$$= \begin{bmatrix} \sigma_1^2 \cdot \Delta t & \rho \sigma_1 \sigma_2 \cdot F(\phi, \Delta t) \\ \rho \sigma_1 \sigma_2 \cdot F(\phi, \Delta t) & \sigma_2^2 \cdot F(2\phi, \Delta t) \end{bmatrix} \quad (12)$$

and the covariance matrix for the measurement equation is assumed to take the form $H = \text{diag}[\sigma_v^2(\tau_1), \dots, \sigma_v^2(\tau_K)]$. That assumption is standard in the literature, as is the assumption that all other contemporaneous and intertemporal covariances are zero.

The starting values for the state variable vector and its covariance matrix $\beta_{1|0} = [0, 0]'$ and:

$$P_{1|0} = \begin{bmatrix} \sigma_\eta^2 / (1 - \psi^2) & \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi} \\ \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi} & \sigma_2^2 \cdot \frac{1}{2\phi} \end{bmatrix} \quad (13)$$

where ψ is the coefficient from the autoregression $\beta_1^*(t + \Delta t) = \psi \cdot \beta_1^*(t) + \eta(t)$ and $\beta_1^*(t)$ is the Level coefficient series obtained from a preliminary estimation of the non-AF two-factor NS model by OLS. These quantities are the unconditional expectations for $\beta(t)$ and its covariance, i.e. $\mathbb{E}[\beta(t)]$ and $\int_0^\infty \Phi(\phi, s) \Omega \Phi(\phi, s) ds$, except for the values purely associated with $\beta_1(t)$. The latter are technically undefined because $\beta_1(t)$ is a unit-root process and so the values from the near-unit-root autoregression are substituted.¹⁴

4.1.3 Econometric issues

It is worthwhile at this stage making several observations about the AF/NS(2) model regarding estimation and identification, given those issues have typically presented challenges for the estimation of latent-factor GATSMs in the past.¹⁵ These observations apply equally to the EA/AF/NS model in the following subsection.

First, the AF/NS(2) model is globally identified. Following the discussion in Collin-Dufresne, Goldstein and Jones (2008) section I, this property is assured because the AF/NS(2) model effectively imposes the restriction that the mean-reversion parameter for the second factor (i.e. $\phi > 0$) will always be greater than for the first factor (i.e. 0). That means the AF/NS(2) model estimation will have a unique maximum and estimated parameter set.

Second, both of the market prices of risk in the AF/NS(2) can be identified from an unrestricted estimation with a sample of zero-coupon data. This property arises because the Level component for the AF/NS(2) model subsumes the constant parameter for the mean level of the short rate in the generic GATSM. Conversely, Singleton (2006) pp. 342-343 notes that the mean short rate parameter and the constant market prices of risk cannot all be identified for a latent-factor GATSM estimated with zero-coupon data.

Third, the implicit econometric identification of the AF/NS(2) model uses $\theta_P = 0$, rather than $\theta_Q = 0$ as suggested in Singleton (2006) pp. 318-319 for the canonical two-LF/GATSM. The practical implication is to leave the constant market prices of risk in the measurement equation (i.e. $\sigma_1 \gamma_{0,1} \cdot \frac{1}{2} \tau$ in the interest rate function) and have no constant in the state equation. That identification is convenient for the empirical application in this article because it matches the implicit identification used in RW.¹⁶ Conversely, the identification $\theta_Q = 0$ used for the AF/NS model in Christensen et al.

(2010) eliminates the constant market prices of risk from the measurement equation and includes a constant in the state equation (which incorporates the constant market price of risk parameters). Nevertheless, both identifications provide observationally-equivalent representations of the data, and are within an invariant affine transformation of each other (see Singleton (2006) pp. 319-321).

Finally, from a practical perspective, standard reparametrizations are employed to ensure trouble-free numerical evaluation of the Kalman filter; i.e. a Cholesky specification to ensure the covariance matrices Q and P always remains positive definite, and $\rho = \omega / (1 + |\omega|)$ to respect the ± 1 range for innovation correlations.

4.2 The EA/AF/NS model

The AF/NS(2) model may readily be extended with an essentially affine specification for the market prices of risk; i.e. $\Gamma(t) = \gamma_0 + \gamma_1 \beta(t)$, where $\gamma_0 = [\gamma_{0,1}, \gamma_{0,2}]'$ and γ_1 is a 2×2 matrix of constants. Following Dai and Singleton (2002) appendix 1, the measurement equation remains the same as for the AF/NS(2) model, and the state equation is modified by the matrix exponential $\exp(-\gamma_1)$ to give:

$$\begin{aligned} \beta(t + \Delta t) &= \Phi(\phi, \Delta t) \exp(-\gamma_1 \Delta t) \beta(t) + \varepsilon(t + \Delta t) \\ &= \exp(-\kappa \Delta t) \beta(t) + \varepsilon(t + \Delta t) \end{aligned} \quad (14)$$

where $\kappa = \text{diag}[0, \phi] + \gamma_1$.

The covariance matrix for the state equation may be evaluated as:

$$\begin{aligned} Q &= \int_0^{\Delta t} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= V \begin{bmatrix} u_{11} \cdot F(2d_1, \Delta t) & u_{12} \cdot F(d_1 + d_2, \Delta t) \\ u_{21} \cdot F(d_1 + d_2, \Delta t) & u_{22} \cdot F(2d_2, \Delta t) \end{bmatrix} V' \end{aligned} \quad (15)$$

where VDV^{-1} is the eigensystem decomposition of κ , $D = \text{diag}[d_1, d_2]$, and the elements u_{ij} are those from $U = V^{-1}\Omega(V^{-1})'$. The covariance matrix for the measurement equations is again assumed to be $H = \text{diag}[\sigma_v^2(\tau_1), \dots, \sigma_v^2(\tau_K)]$.

The starting values for the state variables and their covariance are the unconditional expectations, respectively $\beta_{1|0} = [0, 0]'$ and

$$\begin{aligned} P_{1|0} &= \int_0^{\infty} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= V \begin{bmatrix} u_{11} \frac{1}{2d_1} & u_{12} \frac{1}{d_1 + d_2} \\ u_{21} \frac{1}{d_1 + d_2} & u_{22} \frac{1}{2d_2} \end{bmatrix} V' \end{aligned} \quad (16)$$

The system above is restricted to have positive eigenvalues for κ . From a practical perspective, this can be viewed as an additional estimation restriction to ensure the covariance matrices Q and P always remains positive definite. From a theoretical perspective, the restriction also ensures consistency with the assumption from Dai and Singleton (2002) that the eigenvalues of the generic GATSM are strictly positive (which in turn means the eigenvalues of $\exp(-\kappa \Delta t)$ will be less than 1 and so $\beta(t)$ will be stationary).¹⁷ The restriction is enforced in the estimation as follows.

From Higan (1996) p. 223 a non-symmetric matrix may be written as the sum of a symmetric matrix A_S and an antisymmetric matrix A_K , and A will be positive definite if A_S is positive definite. The latter can be generated with three additional parameters as $A_S = LL' + \text{diag}[0, \phi]$, where L is a 2×2 lower-diagonal matrix. The antisymmetric matrix can be generated with one additional parameter d and then setting $A_{12} = -A_{21} = d$ and $A_{11} = A_{22} = 0$. Directly calculating $\kappa = A_S + A_K$ and its eigenvalues gives:

$$\text{eig} \begin{bmatrix} a & b+d \\ b-d & c \end{bmatrix} = \frac{1}{2}c + \frac{1}{2}a \pm \frac{1}{2}\sqrt{(a-c)^2 - 4d^2 + 4b^2} \quad (17)$$

and a reparametrization:

$$d = \frac{e}{1+|e|} \cdot \frac{1}{2}\sqrt{(a-c)^2 + 4b^2} \quad (18)$$

ensures that d will result in a positive value for the square root operand, therefore guaranteeing real positive eigenvalues for κ .

The interest rate as a function of time to maturity τ will again be composed of the average expected short rate, and a term premium function. However, the essentially affine specification for the market prices of risk means that the term premium function will now vary with state variables. The function for the EA/AF/NS(2) model may be evaluated as:

$$\text{TP}(t, \tau) = \text{TP}(\tau) + \left(\left[1, \frac{1}{\tau}F(\phi, \tau) \right] - [1, 1] [\kappa\tau]^{-1} [I - \exp(-\kappa\tau)] \right) \beta(t) \quad (19)$$

where $\text{TP}(\tau)$ is the expression in equation 8.

5 An application to U.S. yield curve dynamics

This section applies the two-factor NS models developed in the prior section to U.S. yield curve from 1971 to 2010. The main purpose is to illustrate the empirical application of the models and their output. However, the results may also be used to cross-check the changes in U.S. yield curve dynamics already documented in Rudebusch and Wu (2007, hereafter RW) over the period 1971 and 2002 using a similar model (i.e. a bivariate latent-factor GATSM with essentially affine market prices of risk). A mechanical extension applying the two-factor NS models to the data from 2003 onward (albeit with suitable caveats to be noted) may also be used to assess subsequent changes in U.S. yield curve dynamics beyond the end of the RW sample.

Section 5.1 provides an overview of the yield curve data for the 1971 to 2010 period and the three subsamples. Section 5.2 applies the two-factor AF/NS(2) model to the full sample and the three subsamples, and section 5.3 applies the EA/AF/NS(2) model.

5.1 Yield curve data

Figure 1 provides an overview of the U.S. yield curve and its dynamics by plotting the 3-month and 15-year government-risk interest rate data (as detailed further below),

and also the spread between those two rates. The latter is defined opposite to the common convention of a long-maturity rate less a short-maturity rate so it coincides qualitatively with the inverted shape of the Slope function in the NS models. Hence, the troughs in the spread represent periods of easy monetary policy and values above zero represent an inverted yield curve.

[Figure 1 around here]

The full sample period is divided into three samples. Sample A is from November 1971 (the beginning of the 15-year data) to December 1987 (to match the end of sample A from RW, which in turn is chosen to be prior to the tenure of FOMC Chairman Greenspan). Sample B is from January 1988 to December 2002 (to match sample B from RW). Sample C, from January 2003 to June 2010 (the latest data available at the time of the analysis), simply contains the additional data available relative to RW. That said, sample C is arguably a unique period in its own right, because it conveniently begins amid the onset of U.S. deflation concerns in late-2002/early-2003 (e.g. see Billi, 2009 p. 83) and ends with the ultra-easy U.S. monetary policy following the 2007/2008 global financial crisis. However, the results for sample C are reported only out of interest and for comparison to the estimates over samples A and B, and should not be taken as an advocacy to apply NS models over this period. It would be theoretically questionable to represent the nominal yield curve in a low to near-zero interest rate environment with a yield curve model that cannot respect the zero bound for nominal interest rates.

The maturity span of the available yield curve data changes over the full sample, reflecting the longest-maturity bond on issue at any point in time. Sample A uses 3- and 6-month Treasury bill rates (from the Federal Reserve Economic Database on the St Louis Federal Reserve website, converted to a continuously compounding basis) and the 1-, 2-, 3-, 4-, 5-, 7-, 10-, and 15-year continuously compounding zero-coupon government interest rates from the data set described in Gürkaynak, Sack and Wright (2008). Sample B is estimated with data of the same maturity span (hereafter denoted 3-m/15-y) to allow a direct comparison to sample A, and also with the addition of the Gürkaynak et al. (2008) 20- and 30-year data (which became available in July 1981 and November 1985 respectively). Sample C is estimated with data of the latter maturity span (hereafter denoted 3-m/30-y) to allow a direct comparison to sample B. All of the data are month-end rates taken from the original sets of daily data.

5.2 Estimation results for the AF/NS(2) model

The Kalman filter recursion is used to evaluate the log likelihood for the model, and the latter is maximized numerically using the Broyden-Fletcher-Goldfarb-Shanno algorithm as supplied in the “fminunc” function of the Matlab optimization toolbox. The asymptotic standard errors are calculated using the Hessian matrix evaluated at the parameter values that maximize the likelihood function.

Consistent with the discussion on global identification from the previous section, convergence for the AF/NS(2) model estimation was timely and reliable with no apparent sensitivity in the end result to different starting values. Figure 2 illustrates the resulting state variables, i.e. the AF/NS(2) Level and Slope coefficients, for the estimation using the 3-month to 15-year yield curve data over the entire sample. The AF/NS(2) Level and Slope coefficients respectively reflect the level and slope of the

yield curve as represented by the 15-year rate and 3-month less 15-year spread in figure 1, and the model-implied short rate, i.e. $r(t) = \beta_1(t) + \beta_2(t)$, reflects the 3-month rate. Note that $r(t)$ falls materially below zero in sample C, which further suggests that NS models (and so, by implication, GATSMs) are likely to be too simplistic for modeling the yield curve within a low to near-zero nominal interest rate environment (particularly if it were important to strictly respect the zero bound for a given application). The results for the individual sample periods and using the 3-m/30-y data are very similar to figure 2, and so are not reported separately.

[Figure 2 around here]

Table 2 contains the estimated parameter values for the AF/NS(2) model using the 3-m/15-y data over the joint sample A+B and the individual samples A and B.¹⁸ Table 3 contains the estimated parameter values for the AF/NS(2) model using the 3-m/30-y data over the joint sample B+C and the individual samples B and C.

[Tables 2 and 3 around here]

The first aspect of note is that the hypotheses of no change in the yield curve DGP between samples A and B, and samples B and C are soundly rejected, with a likelihood ratio statistics of 910.5 and 3532.4 respectively. Figure 3 illustrates that one of the main points of difference between the two samples is that the (time invariant) term premium function is smaller in sample B than in sample A. That in turn mainly reflects a lower market price of risk for the Slope component, and also lower volatilities for the Level and Slope components. The term premium function falls again in sample C, mainly due to a further fall in the market price of risk for the Slope component.

[Figure 3 around here]

Other points of note for the AF/NS(2) model estimates are the material decline in the mean-reversion parameter ϕ for the Slope coefficient from sample A to B, and the sign reversal of the innovation correlation parameter ρ from sample B to C. In practical terms, the latter suggests that positive innovations to the Slope coefficient (e.g. unanticipated policy tightenings) were on average associated with negative innovations to the Level coefficient (i.e. a fall in long-maturity yields) over sample C.

5.3 Estimation results for the EA/AF/NS(2) model

Convergence for the model estimation via the Kalman filter was again timely and reliable, with no apparent sensitivity in the end result to the different starting values tested for the model parameters. All estimates of the Level and Slope coefficients for the EA/AF/NS(2) model were again very similar to figure 2 and so are not separately reported.

Tables 4 and 5 contain the estimated parameter values for the EA/AF/NS(2) model over the combined and individual samples. The hypotheses of no change in yield curve dynamics between the samples A and B, and the samples B and C are again soundly rejected by the likelihood ratio tests.

Figures 4 and 5 illustrate the term premium estimates for the time to maturity of five years for each of the different sample estimated. These are obtained using the relevant estimated parameters and state variables in equation 19 with $\tau = 5$.

[Tables 4 and 5 around here]

Figures 4 and 5 confirm that changes in the average level of the term premium estimates are again a major point of difference between the individual samples, although

the more complex specification for the term premium function in the EA/AF/NS(2) model makes it harder to attribute differences to individual parameters. A more transparent alternative is simply to attribute the 5-year term premia estimates to their time-invariant and time-varying components. Evaluating the point estimates of the time-invariant component $TP(\tau)$ of the term premium function for the EA/AF/NS(2) model confirms the AF/NS(2) pattern of results; i.e. there is still a declining time-invariant term premium component from sample A to sample C (except sample B based on 3-m/15-y data, which is arguably less reliable).¹⁹ Nevertheless, the average term premium during sample A is lower than sample B, meaning that the time-varying component of the 5-year term premium was lower in sample A than sample B. Regarding variation in term premia, sample B (based on 3-m/30-y data) shows the least variation in terms of the peak-to-trough range and the standard deviation of first differences. Sample C shows the most variation.

[Figures 4 and 5 around here]

Other points of note for the EA/AF/NS(2) model estimates are the confirmation of the material decline in the mean-reversion parameter ϕ and the sign reversal of the innovation correlation parameter ρ as already discussed for the AF/NS(2) model. Indeed, the model estimate suggests that the innovations become all but perfectly negative, albeit with an implausibly large confidence interval. That suggests some degree of overparametrization when applying the EA/AF/NS(2) model over the relatively short sample period and/or within a low to near-zero nominal interest rate environment.

5.4 Model comparisons

The significance of the time-varying component of the term premium estimates can be assessed with a likelihood ratio test given that the EA/AF/NS(2) model nests the AF/NS(2) model; i.e. the latter is the EA/AF/NS(2) model with $\gamma_1 = 0$ in $\Gamma(t) = \gamma_0 + \gamma_1\beta(t)$. The likelihood ratio tests in tables 3 and 4 show that the estimates of γ_1 are typically highly significant, with an exception being sample B using the 3-m/30-y data. These results imply that term premia are better modeled as time-varying rather than constant over the sample, except sample B.

When comparing the EA/AF/NS(2) model application to the RW two-LF/GATSM application, it is first worth noting that the model specification and the estimation approach differs in several respects.²⁰ First, RW sets to zero the constant element of the market price of risk for the persistent component parameter, which allows the identification of the mean parameter for the short rate. Conversely, the EA/AF/NS(2) model allows all market price of risk parameters to be identified, and the results are material and statistically significant. Second, RW specifies a diagonal innovation covariance matrix for the state variables. Conversely, the EA/AF/NS(2) model allows for correlated innovations, and the correlation is found to be material, but not usually statistically significant. Third, RW restricts insignificant parameters for the remaining market price of risk specification (from an initial estimation) to zero, while the EA/AF/NS(2) model retains all of the market price of risk parameters. The estimate of the γ_1 matrix for EA/AF/NS(2) model over the sample A+B indicates how that initial zeroing of insignificant risk parameters could adversely affect the final model. That is, all of the individual estimates of the γ_1 matrix elements are insignificant, but γ_1 is significant as a whole.

Regarding the empirical results for the EA/AF/NS(2) model, they confirm the main finding in RW that a statistically significant change in term structure behavior occurred between samples A and B. The pattern of term premium estimates over the two samples is also similar to that of RW, with a fall in the variation of the 5-year term premium estimates from sample A to sample B. RW removes the sample averages from the term premium estimates, and so the absolute levels cannot be compared.

6 Conclusion

This article establishes that most NS models may be obtained as optimal approximations to the dynamic component of the generic GATSM outlined in Dai and Singleton (2002). That theoretical foundation provides an assurance that NS models correspond to a well-accepted framework for yield curve modeling, and it also motivates the development and application of a two-factor arbitrage-free NS model as a standard model of the yield curve. That is, regardless of the true number of factors, their nature (e.g. economic, financial, etc.) and the potential complexity of their interactions, the Level component of the NS model will represent the persistent components of any GATSM model, and the Slope component will represent the non-persistent components.

As a practical illustration of applying an NS model, this article develops a two-factor arbitrage-free NS model and uses it to test for changes in U.S. yield dynamics. The results from applying the NS model confirm the main findings of Rudebusch and Wu (2007): there was a very material change in the data-generating process for the U.S. yield curve between the sample from 1971 to 1988 and the sample from 1988 to 2002. An additional estimation of the NS model using data from 2003 to 2010 indicates a further material change in U.S. yield curve behavior.

A CIR dynamics

This appendix shows by example that dynamic term structure models with Cox et al. (1985b)/square-root innovations cannot be optimally approximated using NS factor loadings, in the sense of following a procedure analogous to the exposition in section 3.

Assume N independent factors each with the form $dX_n(t) = \kappa_n [\theta_n - X_n(t)] dt + \sigma_n \sqrt{X_n(t)} dW(t)$ under the risk-neutral Q measure.

Then $P(t, T) = \exp \left[\sum_{n=1}^N A_n(t, T) + B_n(t, T) X_n(t) \right]$ where each $B_n(t, T)$ has the standard Cox et al. (1985b) form:²¹

$$B_n(t, T) = \frac{2[1 - \exp(\gamma_n \tau)]}{(\gamma_n + \kappa_n)[\exp(\gamma_n \tau) - 1] + 2\gamma_n} \quad (20)$$

with $\gamma_n = \sqrt{\kappa_n^2 + 2\sigma_n^2}$.

The associated forward rate curve is:

$$f(t, T) = a_0 + \sum_{n=1}^N \frac{4\gamma_n^2 \exp(\gamma_n \tau)}{[(\gamma_n + \kappa_n)[\exp(\gamma_n \tau) - 1] + 2\gamma_n]^2} X_n(t) - \frac{\partial}{\partial \tau} A_n(\tau) \quad (21)$$

The relative complexity of this functional form of maturity means that a central exponential decay term $\exp(-\phi\tau)$ cannot be factored out of each factor loading as for the Gaussian case in section 3.

This incompatibility of the NS class of yield curve models with CIR/square-root dynamics is unfortunate, because CIR models have the well-known advantage over GATSMs of respecting the zero bound for interest rates. One resulting implication is that, in cases where the probability of zero interest rates from Gaussian dynamics is material, a non-Gaussian dynamic term structure model might be more appropriate than an NS model. That caveat applies in particular if the application requires the zero bound to be strictly respected (e.g. for financial market applications such as option pricing).

B Details of calculations for section 5 [NOT INTENDED FOR PUBLICATION]

B.1 AF/NS(2) forward rate curve

B.1.1 AF/NS(2) forward rate expression

Equation 5 from the main text expresses the AF/NS(2) forward rate in terms of three components: (1) the expected path of the short rate component, the term premium component, and the volatility effect component. These are evaluated respectively in the subsections below.

B.1.2 The expected path of the short rate component

$$\begin{aligned}\mathbb{E}_t[r(t+\tau)] &= \beta_1(t) + \beta_2(t) \cdot \exp(-\phi\tau) \\ &= [1, \exp(-\phi\tau)] \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} \\ &= b(\tau)\beta(t)\end{aligned}$$

where $b(\tau) = [1, \exp(-\phi\tau)]$ and $\beta(t) = [\beta_1(t), \beta_2(t)]'$.

B.1.3 Term premium component

$$\int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi s)] \begin{bmatrix} \gamma_{0,1} \\ \gamma_{0,2} \end{bmatrix} ds = \sigma_1 \gamma_{0,1} \int_0^\tau ds + \sigma_2 \gamma_{0,2} \int_0^\tau \exp(-\phi s) ds$$

The Level and Slope term premium components are evaluated respectively in the subsections below.

Term premium component for Level

$$\begin{aligned}\sigma_1 \gamma_{0,1} \int_0^\tau ds &= \sigma_1 \gamma_{0,1} (s|_0^\tau) \\ &= \sigma_1 \gamma_{0,1} \cdot \tau\end{aligned}$$

Term premium component for Slope

$$\begin{aligned}\int_0^\tau \sigma_2 \gamma_{0,2} \exp(-\phi[\tau-s]) ds &= \sigma_2 \gamma_{0,2} \frac{1}{\phi} \exp(-\phi[\tau-s]) \Big|_0^\tau \\ &= \sigma_2 \gamma_{0,2} \cdot \frac{1}{\phi} [1 - \exp(-\phi\tau)] \\ &= \sigma_2 \gamma_{0,2} \cdot F(\phi, \tau)\end{aligned}$$

where $F(\phi, \tau) = \frac{1}{\phi} [1 - \exp(-\phi\tau)]$.

B.1.4 Volatility effect component

$$\begin{aligned}
& \int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi[\tau-s])] \left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \int_s^\tau \begin{bmatrix} \sigma_1 \\ \sigma_2 \exp(-\phi[u-s]) \end{bmatrix} du \right) ds \\
= & \int_0^\tau [\sigma_1, \sigma_2 \exp(-\phi[\tau-s])] \begin{bmatrix} \sigma_1 \int_s^\tau du + \rho \sigma_2 \int_s^\tau \exp(-\phi[u-s]) du \\ \rho \sigma_1 \int_s^\tau du + \sigma_2 \int_s^\tau \exp(-\phi[u-s]) du \end{bmatrix} ds \\
= & \sigma_1^2 \int_0^\tau \left(\int_s^\tau du \right) ds \\
& + \rho \sigma_1 \sigma_2 \int_0^\tau \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds \\
& + \rho \sigma_1 \sigma_2 \int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau du \right) ds \\
& + \sigma_2^2 \int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds
\end{aligned}$$

where the last four lines are, respectively, the Level/Level, Level/Slope, Slope/Level, and Slope/Slope components. These are evaluated in turn in the subsections below.

Volatility effect component for Level/Level

$$\begin{aligned}
\int_0^\tau \left(\int_s^\tau du \right) ds &= \int_0^\tau (u|_s^\tau) ds \\
&= \int_0^\tau (\tau - s) ds \\
&= \left(\tau s - \frac{1}{2} s^2 \right) \Big|_0^\tau \\
&= \frac{1}{2} \tau^2
\end{aligned}$$

Volatility effect component for Level/Slope

$$\begin{aligned}
\int_0^\tau \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds &= \int_0^\tau \left(-\frac{1}{\phi} \exp(-\phi[u-s]) \Big|_s^\tau \right) ds \\
&= \frac{1}{\phi} \int_0^\tau (1 - \exp(-\phi[\tau-s])) ds \\
&= \frac{1}{\phi} \left(s - \frac{1}{\phi} \exp(-\phi[\tau-s]) \right) \Big|_0^\tau \\
&= \frac{\tau}{\phi} - \frac{1}{\phi^2} + \frac{1}{\phi^2} \exp(-\phi\tau)
\end{aligned}$$

Volatility effect component for Slope/Level

$$\begin{aligned}
\int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau du \right) ds &= \int_0^\tau \exp(-\phi[\tau-s]) (\tau-s) ds \\
&= \left(\exp(-\phi[\tau-s]) \frac{1}{\phi^2} (1 + \tau\phi - s\phi) \right) \Big|_0^\tau \\
&= \frac{1}{\phi^2} - \frac{1}{\phi^2} \exp(-\phi\tau) - \frac{\tau}{\phi} \exp(-\phi\tau)
\end{aligned}$$

Volatility effect component for Level/Slope + Slope/Level

$$\begin{aligned}
\int_0^\tau \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds + \int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau du \right) ds &= \frac{\tau}{\phi} - \frac{\tau}{\phi} \exp(-\phi\tau) \\
&= \tau \cdot F(\phi, \tau)
\end{aligned}$$

Volatility effect component for Slope/Slope

$$\begin{aligned}
&\int_0^\tau \exp(-\phi[\tau-s]) \left(\int_s^\tau \exp(-\phi[u-s]) du \right) ds \\
&= \int_0^\tau \exp(-\phi s) \left(\frac{1}{\phi} (\exp(-\phi[\tau-s]) - 1) \right) ds \\
&= \frac{1}{\phi^2} \left(\exp(2\phi\tau) \left(\exp(-\phi[\tau-s]) - \frac{1}{2} \exp(-2\phi s) \right) \right) \Big|_0^\tau \\
&= \frac{1}{2\phi^2} [1 - \exp(-\phi\tau)]^2
\end{aligned}$$

B.1.5 AF/NS(2) forward rate curve

Substituting the results above into the AF/NS(2) forward rate expression gives equation 6.

B.2 AF/NS(2) interest rate curve

The AF/NS(2) interest rate curve may be evaluated using the standard relationship $R(t, \tau) = \frac{1}{\tau} \int_0^\tau f(t, \tau) d\tau$ for each component of the AF/NS(2) forward rate curve. The calculations are undertaken respectively in the subsections below.

B.2.1 Short rate Level component

$$\begin{aligned}
\frac{1}{\tau} \int_0^\tau d\tau &= \frac{1}{\tau} (\tau|_0^\tau) \\
&= \frac{1}{\tau} (\tau) \\
&= 1
\end{aligned}$$

B.2.2 Short rate Slope component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau \exp(-\phi\tau) d\tau &= \frac{1}{\tau} \left(-\frac{1}{\phi} \exp(-\phi\tau) \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left(\frac{1}{\phi} [1 - \exp(-\phi\tau)] \right) \\ &= \frac{1}{\tau} F(\phi, \tau)\end{aligned}$$

B.2.3 Term premium Level component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau \tau d\tau &= \frac{1}{\tau} \left(\frac{1}{2} \tau^2 \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left(\frac{1}{2} \tau^2 \right) \\ &= \frac{1}{2}\end{aligned}$$

B.2.4 Term premium Slope component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau F(\phi, \tau) d\tau &= \frac{1}{\tau} \int_0^\tau \frac{1}{\phi} [1 - \exp(-\phi\tau)] d\tau \\ &= \frac{1}{\tau} \left(\frac{\tau}{\phi} + \frac{1}{\phi^2} \exp(-\phi\tau) \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left[\frac{\tau}{\phi} - \frac{1}{\phi^2} + \frac{1}{\phi^2} \exp(-\phi\tau) \right] \\ &= \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right]\end{aligned}$$

B.2.5 Volatility effect Level/Level component

$$\begin{aligned}\frac{1}{\tau} \int_0^\tau \frac{1}{2} \tau^2 d\tau &= \frac{1}{\tau} \left(\frac{1}{6} \tau^3 \Big|_0^\tau \right) \\ &= \frac{1}{\tau} \left(\frac{1}{6} \tau^3 \right) \\ &= \frac{1}{6}\end{aligned}$$

B.2.6 Volatility effect Slope/Slope component

$$\begin{aligned}
\frac{1}{\tau} \int_0^\tau [F(\phi, \tau)]^2 d\tau &= \frac{1}{\tau} \int_0^\tau \frac{1}{\phi^2} [1 - \exp(-\phi\tau)]^2 d\tau \\
&= \frac{1}{\tau} \left(\frac{1}{2\phi^2} \tau + \frac{1}{\phi^3} \exp(-\phi\tau) - \frac{1}{4\phi^3} \exp(-2\phi\tau) \Big|_0^\tau \right) \\
&= \frac{1}{\tau} \left[\frac{\tau}{2\phi^2} - \frac{3}{4\phi^3} + \frac{1}{\phi^3} \exp(-\phi\tau) - \frac{1}{4\phi^3} \exp(-2\phi\tau) \right] \\
&= \frac{1}{2\phi^2} - \frac{1}{2\phi^3\tau} [1 - \exp(-\phi\tau)] - \frac{1}{4\phi^3\tau} [1 - \exp(-\phi\tau)]^2 \\
&= \frac{1}{2\phi^2} \left(1 - \frac{1}{\tau} F(\phi, \tau) - \frac{1}{2\tau} \phi [F(\phi, \tau)]^2 \right)
\end{aligned}$$

B.2.7 Volatility effect Level/Slope + Slope/Level component

$$\begin{aligned}
\frac{1}{\tau} \int_0^\tau \tau F(\phi, \tau) d\tau &= \frac{1}{\tau} \int_0^\tau \frac{\tau}{\phi} [1 - \exp(-\phi\tau)] d\tau \\
&= \frac{1}{\tau} \left(\frac{1}{2\phi} \tau^2 + \frac{1}{\phi^3} \exp(-\phi\tau) + \frac{\tau}{\phi^2} \exp(-\phi\tau) \Big|_0^\tau \right) \\
&= \frac{1}{\tau} \left(\frac{1}{2\phi} \tau^2 - \frac{1}{\phi^3} + \frac{1}{\phi^3} \exp(-\phi\tau) + \frac{\tau}{\phi^2} \exp(-\phi\tau) \right) \\
&= \frac{1}{2\phi} \tau - \frac{1}{\phi^2\tau} F(\phi, \tau) + \frac{1}{\phi^2} - \frac{1}{\phi^2} \exp(-\phi\tau) \\
&= \frac{1}{\phi^2} \left(1 - \frac{1}{\tau} F(\phi, \tau) + \frac{1}{2} \phi\tau - \frac{1}{\phi^2\tau} F(\phi, \tau) \right)
\end{aligned}$$

B.2.8 Final interest rate curve expression

$$\begin{aligned}
R(t, \tau) &= \beta_1(t) + \beta_2(t) \cdot \frac{1}{\tau} F(\phi, \tau) \\
&\quad + \sigma_1 \gamma_1 \cdot \frac{1}{2} \tau + \sigma_2 \gamma_2 \cdot \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right] \\
&\quad - \sigma_1^2 \cdot \frac{1}{6} \tau^2 - \sigma_2^2 \cdot \frac{1}{2\phi^2} \left[1 - \frac{1}{\tau} F(\phi, \tau) - \frac{1}{2\tau} \phi [F(\phi, \tau)]^2 \right] \\
&\quad - \rho \sigma_1 \sigma_2 \cdot \frac{1}{\phi^2} \left[1 - \frac{1}{\tau} F(\phi, \tau) + \frac{1}{2} \phi\tau - \frac{1}{\phi^2\tau} F(\phi, \tau) \right]
\end{aligned}$$

which may be expressed in the form $R(t, \tau) = \bar{a}(\tau) + \bar{b}(\tau) \beta(t)$ as in the main text.

B.3 AF/NS(2) model Kalman filter calculations

B.3.1 State equation

The AF/NS(2) state equation may be derived directly from the expected path of the short rate, i.e.:

$$\begin{aligned}
\mathbb{E}_t \{ \mathbb{E}_{t+\tau} r(t + \tau + \Delta t) \} &= \mathbb{E}_t [r(t + \tau + \Delta t)] \\
\mathbb{E}_t \{ [1, \exp(-\phi\tau)] \beta(t + \Delta t) \} &= [1, \exp(-\phi[\tau + \Delta t])] \beta(t) \\
[1, \exp(-\phi\tau)] \mathbb{E}_t \{ \beta(t + \Delta t) \} &= [1, \exp(-\phi\tau)] \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\phi\Delta t) \end{bmatrix} \beta(t) \\
\mathbb{E}_t \{ \beta(t + \Delta t) \} &= \Phi(\phi, \Delta t) \beta(t)
\end{aligned}$$

Removing the expectations operator, the state equation is therefore:

$$\beta(t + \Delta t) = \Phi(\phi, \Delta t) \beta(t) + \varepsilon(t + \Delta t)$$

as in equation 8, where $\mathbb{E}_t [\varepsilon(t + \Delta t)] = 0$. Note that $\varepsilon(t + \Delta t)$ has a correlated bivariate Gaussian distribution given the correlated innovations assumed to underlie equation 5, i.e. $\Omega [dW_1(t), \exp(-\phi\tau) \cdot dW_2(t)]$.

B.3.2 State covariance matrix

$$\begin{aligned}
Q &= \int_0^{\Delta t} \Phi(\phi, s) \Omega [\Phi(\phi, s)]' ds \\
&= \int_0^{\Delta t} \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\phi s) \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\phi s) \end{bmatrix} ds \\
&= \int_0^{\Delta t} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \exp(-\phi s) \\ \rho\sigma_1\sigma_2 \exp(-\phi s) & \sigma_2^2 \exp(-2\phi s) \end{bmatrix} ds \\
&= \left[\begin{array}{cc} \sigma_1^2 s & -\rho\sigma_1\sigma_2 \frac{1}{\phi} \exp(-\phi s) \\ -\rho\sigma_1\sigma_2 \exp(-\phi s) & -\sigma_2^2 \frac{1}{2\phi} \exp(-2\phi s) \end{array} \right] \Big|_0^{\Delta t} \\
&= \begin{bmatrix} \sigma_1^2 \Delta t & \rho\sigma_1\sigma_2 \left[\frac{1}{\phi} - \frac{1}{\phi} \exp(-\phi \Delta t) \right] \\ \rho\sigma_1\sigma_2 \left[\frac{1}{\phi} - \frac{1}{\phi} \exp(-\phi \Delta t) \right] & \sigma_2^2 \frac{1}{2\phi} [1 - \exp(-2\phi \Delta t)] \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 \cdot \Delta t & \rho\sigma_1\sigma_2 \cdot F(\phi, \Delta t) \\ \rho\sigma_1\sigma_2 \cdot F(\phi, \Delta t) & \sigma_2^2 \cdot F(2\phi, \Delta t) \end{bmatrix}
\end{aligned}$$

B.3.3 Unconditional state covariance matrix

$$\begin{aligned}
P_{1|0} &= \int_0^{\infty} \Phi(\phi, s) \Omega [\Phi(\phi, s)]' ds \\
&= \left[\begin{array}{cc} \sigma_1^2 s & -\rho\sigma_1\sigma_2 \frac{1}{\phi} \exp(-\phi s) \\ -\rho\sigma_1\sigma_2 \exp(-\phi s) & -\sigma_2^2 \frac{1}{2\phi} \exp(-2\phi s) \end{array} \right] \Big|_0^{\infty} \\
&= \begin{bmatrix} \text{undef} & \rho\sigma_1\sigma_2 \frac{1}{\phi} \\ \rho\sigma_1\sigma_2 \frac{1}{\phi} & \sigma_2^2 \frac{1}{2\phi} \end{bmatrix}
\end{aligned}$$

where the indefinite integral evaluations use the results from the previous subsection. Note that $P_{1|0}(1, 1) = \sigma_\eta^2 / (1 - \psi^2)$ replaces the undefined expression “undef” to give $P_{1|0}$ as in equation 12.

B.3.4 Cholesky specification for Ω

$$\begin{aligned} LL' &= \Omega \\ \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \rho^2\sigma_2^2 + \sigma_2^2(-\rho^2 + 1) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \end{aligned}$$

B.4 AF/NS(2) term premium functions

The AF/NS(2) forward term premium function can be obtained by subtracting the AF/NS(2) forward rate expression excluding term premia (i.e. by setting $\gamma_0 = 0$) from the AF/NS(2) forward rate expression with term premia, i.e.:

$$\begin{aligned} \text{FTP}(\tau) &= f(t, \tau) - [f(t, \tau) | \gamma_0 = 0] \\ &= \sigma_1\gamma_{0,1} \cdot \tau + \sigma_2\gamma_{0,2} \cdot F(\phi, \tau) \end{aligned}$$

The interest rate term premium function is then:

$$\begin{aligned} \text{TP}(\tau) &= \frac{1}{\tau} \int_0^\tau \text{FTP}(\tau) d\tau \\ &= \sigma_1\gamma_1 \cdot \frac{1}{2}\tau + \sigma_2\gamma_2 \cdot \frac{1}{\phi} \left[1 - \frac{1}{\tau} F(\phi, \tau) \right] \end{aligned}$$

B.5 The EA/AF/NS(2) model

There is almost certainly a straightforward way to establish the expressions for the EA/AF/NS(2) model using a suitable modification to the HJM framework. But, assuming that route exists, it has unfortunately escaped the investigations of the author.

However, it turns out that the EA/AF/NS(2) model is precisely a special case of the Dai and Singleton (2002, hereafter DS) generic GATSM, as introduced in section 2, with two factors and limit of zero for one of the rates of mean reversion. The derivations in the following subsections are therefore based on the generic GATSM with the substitution of the EA/AF/NS(2) model parameters; i.e. $X(t) = \beta(t)$, $\xi_0 = 0$, $\xi_1 = [1, 1]'$, $K_Q = \text{diag}[0, \phi]$, $\pi_0 = -\gamma_0$, $\pi_1 = -\gamma_1$, and $K_P = \kappa = \text{diag}[0, \phi] + \gamma_1$.

Regarding the EA/AF/NS(2) forward rate and interest rate curves, the time-invariant term premia associated with the constant market prices of risk γ_0 remain via the identification $\theta_P = 0$, as discussed at the end of section 6.2.1. The time-varying term premia associated with matrix γ_1 are captured in the state equation, as discussed in the following subsection. The measurement equation for the EA/AF/NS(2) model therefore remains identical to that of the AF/NS(2) model.

B.6 EA/AF/NS(2) model Kalman filter calculations

B.6.1 State equation

DS appendix A gives the following expression for the generic GATSM state equation:

$$\mathbb{E}_t [X(t + \Delta t)] = \exp(-K_P \tau) X(t) + [I - \exp(-K_P \tau)] \theta_P$$

Substituting the EA/AF/NS(2) parameters from the previous subsection into the generic GATSM state equation expression gives $\exp(-K_Q \Delta t) = \text{diag}[1, \exp(-\phi \Delta t)] = \Phi(\phi, \Delta t)$, $\exp(\pi_1 \tau) = \exp(-\gamma_1 \tau)$, and $\exp(-K_P \Delta t) = \exp(-\kappa \Delta t)$. Therefore the EA/AF/NS(2) state equation is:

$$\begin{aligned} \mathbb{E}_t [\beta(t + \Delta t)] &= \Phi(\phi, \Delta t) \exp(-\gamma_1 \tau) \beta(t) \\ &= \exp(-\kappa \tau) \beta(t) \end{aligned}$$

as in equation 14. Note that $\varepsilon(t + \Delta t)$ is Gaussian as for the AF/NS(2) model.

B.6.2 State covariance matrix

$$\begin{aligned} Q &= \int_0^{\Delta t} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= \int_0^{\Delta t} \exp(-VDV^{-1}s) \Omega [\exp(-VDV^{-1}s)]' ds \\ &= \int_0^{\Delta t} V \exp(-Ds) V^{-1} \Omega [\exp(-VDV^{-1}s)]' ds \\ &= \int_0^{\Delta t} V \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} V^{-1} \Omega (V^{-1})' \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} V' ds \\ &= V \left(\int_0^{\Delta t} \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \exp(-d_1 s) & 0 \\ 0 & \exp(-d_2 s) \end{bmatrix} ds \right) V' \\ &= V \left(\int_0^{\Delta t} \begin{bmatrix} u_{11} \exp(-2d_1 s) & u_{12} \exp(-[d_1 + d_2] s) \\ u_{21} \exp(-[d_1 + d_2] s) & u_{22} \exp(-2d_2 s) \end{bmatrix} ds \right) V' \\ &= V \begin{bmatrix} u_{11} \cdot F(2d_1, \Delta t) & u_{12} \cdot F(d_1 + d_2, \Delta t) \\ u_{21} \cdot F(d_1 + d_2, \Delta t) & u_{22} \cdot F(2d_2, \Delta t) \end{bmatrix} V' \end{aligned}$$

B.6.3 Unconditional state covariance matrix

$$\begin{aligned} P_{1|0} &= \int_0^{\infty} \exp(-\kappa s) \Omega \exp(-\kappa' s) ds \\ &= V \begin{bmatrix} u_{11} F_2(2d_1, \infty) & u_{12} F_2(d_1 + d_2, \infty) \\ u_{21} F_2(d_1 + d_2, \infty) & u_{22} F_2(2d_2, \infty) \end{bmatrix} V' \\ &= V \begin{bmatrix} u_{11} \frac{1}{2d_1} & u_{12} \frac{1}{d_1 + d_2} \\ u_{21} \frac{1}{d_1 + d_2} & u_{22} \frac{1}{2d_2} \end{bmatrix} V' \end{aligned}$$

where the integral evaluations use the results from the previous subsection.

B.7 EA/AF/NS(2) term premium functions

The time-varying component of the EA/AF/NS(2) forward term premium function may be obtained using the time-varying component of the DS forward term premium expression for the generic GATSM model.

B.7.1 DS discrete-time expression for forward term premium

DS appendix A obtains the following generic GATSM forward term premium expression p_t^n using discrete increments of time Δt :

$$p_t^n = f_t^n - \mathbb{E}_t[r_{t+n\Delta t}]$$

where:

$$f_t^n = -\frac{1}{\Delta} \log \frac{P(t, [n+1]\Delta t)}{P(t, n\Delta t)}$$

and:

$$\begin{aligned} \mathbb{E}_t[r_{t+n\Delta t}] &= \mu_n + \nu_n' X(t) \\ \mu_n &= a_1 + \theta_P' [I - \exp(-K_P' n\Delta t)] b_1 \\ \nu_n &= \exp(-K_P' n\Delta t) b_1 \end{aligned}$$

Note that a_1 and b_1 are constants associated with the one-period interest rate $r_t = a_1 + b_1 X(t)$. DS also notes that f_t^n is the one-period forward deliverable n -periods forward, and $\mathbb{E}_t[r_{t+n\Delta t}]$ is the conditional mean of the short rate.

B.7.2 DS continuous-time expression for forward term premium

In the reverse order of which they were introduced, take the limit of each quantity in the previous section as $\Delta t \rightarrow 0$. Hence, $\lim_{\Delta t \rightarrow 0} r_t = r(t) = \xi_0 + \xi_1' X(t)$ as defined in section 2, so $\lim_{\Delta t \rightarrow 0} b_1 = \xi_1$ and $\lim_{\Delta t \rightarrow 0} a_1 = \xi_0$. Therefore:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \nu_n &= \nu(\tau) = \exp(-K_P' \tau) \xi_1 \\ \lim_{\Delta t \rightarrow 0} \mu_n &= \mu(\tau) = \xi_0 + \theta_P' [I - \exp(-K_P' \tau)] \xi_0 \\ \lim_{\Delta t \rightarrow 0} \mathbb{E}_t[r_{t+n\Delta t}] &= \xi_0 + \theta_P' [I - \exp(-K_P' \tau)] \xi_0 + \xi_1' \exp(-K_P \tau) X(t) \end{aligned}$$

Regarding the forward rate:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} f_t^n &= -\frac{\partial}{\partial T} \log P(t, T) = f(t, T) \\ &= \xi_0 + [\exp(-K_Q' \tau) \xi_1]' X(t) - \frac{\partial}{\partial \tau} A(\tau) \end{aligned}$$

where $P(t, T)$ and $f(t, T)$ are as outlined in section 2.

The forward term premium expression therefore becomes:

$$\begin{aligned}
\lim_{\Delta t \rightarrow 0} p_t^n &= p(t, \tau) \\
&= f(t, T) - \mathbb{E}_t[r_{t+n\Delta t}] \\
&= \xi_0 + \xi'_1 \exp(-K_Q \tau) X(t) - \frac{\partial}{\partial \tau} A(\tau) \\
&\quad - \xi_0 - \theta'_P [I - \exp(-K'_P \tau)] \xi_0 - \xi'_1 \exp(-K_P \tau) X(t)
\end{aligned}$$

and the time-varying component of $p(t, \tau)$ is:

$$p(t, \tau) - p(\tau) = \xi'_1 \exp(-K_Q \tau) X(t) - \xi'_1 \exp(-K_P \tau) X(t)$$

B.7.3 EA/AF/NS(2) forward term premium function

Substituting the EA/AF/NS(2) parameters and expressions from section B.5 and B.6.1 into the time-varying component of $p(t, \tau)$ from the previous section gives:

$$\begin{aligned}
p(t, \tau) - p(\tau) &= \{[1, 1] \Phi(\phi, \Delta t)\} \beta(t) - [1, 1] \exp(-\kappa \tau) \beta(t) \\
\text{FTP}(t, \tau) - \text{FTP}(\tau) &= [1, \exp(-\phi \Delta t)] \beta(t) - [1, 1] \exp(-\kappa \tau) \beta(t)
\end{aligned}$$

B.7.4 EA/AF/NS(2) interest rate term premium function

$$\begin{aligned}
\text{TP}(t, \tau) &= \frac{1}{\tau} \int_0^\tau \text{FTP}(t, \tau) d\tau \\
&= \text{TP}(\tau) + \left(\frac{1}{\tau} \int_0^\tau [1, \exp(-\phi \Delta t)] d\tau \right) \beta(t) - [1, 1] \frac{1}{\tau} \left(\int_0^\tau \exp(-\kappa \tau) d\tau \right) \beta(t) \\
&= \text{TP}(\tau) + \left[1, \frac{1}{\tau} F(\phi, \tau) \right] \beta(t) - [1, 1] \left(\frac{1}{\tau} \left\{ -[\kappa]^{-1} \exp(-\kappa \tau) \Big|_0^\tau \right\} \right) \beta(t) \\
&= \text{TP}(\tau) + \left[1, \frac{1}{\tau} F(\phi, \tau) \right] \beta(t) - [1, 1] [\kappa \tau]^{-1} [I - \exp(-\kappa \tau)] \beta(t) \\
&= \text{TP}(\tau) + \left(\left[1, \frac{1}{\tau} F(\phi, \tau) \right] - [1, 1] [\kappa \tau]^{-1} [I - \exp(-\kappa \tau)] \right) \beta(t)
\end{aligned}$$

Figure legends

Figure 1:

The 3-month and 15-year continuously compounding zero-coupon interest rate data and the spread between those two rates.

Figure 2:

The estimated Level and Slope coefficients and the model-implied short rate, i.e. $\beta_1(t)$, $\beta_2(t)$, and $r(t) = \beta_1(t) + \beta_2(t)$, for the AF/NS(2) model estimated over the full sample using the 3-m/15-y data.

Figure 3:

Term premium functions as implied by the point estimates of the AF/NS(2) model parameters in each sample, using the 3-m/15-y data and/or 3-m/30-y data as indicated.

Figure 4:

The 5-year interest rate data and the 5-year term premium implied by the point estimates of the EA/AF/NS(2) model parameters and state variables in each sample. Sample A+B uses the 3-m/15-y data and sample B+C uses 3-m/30-y data, as indicated.

Figure 5:

The 5-year interest rate data and the 5-year term premium implied by the point estimates of the EA/AF/NS(2) model parameters and state variables in each sample, using the 3-m/15-y data and/or 3-m/30-y data as indicated.

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Notes

¹Bank for International Settlements (2005) provides an overview of routine central bank use of NS models and Coroneo, Nyholm and Vidova-Koleva (2008) notes their widespread use by financial market practitioners. Diebold, Piazzesi and Rudebusch (2005) summarizes some recent time series applications and more examples of NS model applications are referenced later in the present article.

²For example, from Christensen, Diebold and Rudebusch (2010) footnote 4: “Our strategy is to find the affine AF model with factor loadings that match Nelson-Siegel exactly.”

³Dahlquist and Svensson (1996) originally advocated the suitability of NS models for monetary policy purposes on these grounds. Diebold and Li (2006) extends the discussion to a dynamic setting. Besides the examples noted in the present article, a more extensive list of applications is contained in the working paper version.

⁴While Duffie and Singleton (1999) focuses on default risk, pp. 193-94 of that article notes that other financial factors may also be treated in a similar manner.

⁵For example, all of the macrofinance models summarized in Rudebusch (2010) are specified with Gaussian innovations.

⁶Any other central measure of $(\lambda_{n_0+1}, \dots, \lambda_N)$ would suffice for the exposition in this article. In practice, ϕ is an estimated parameter.

⁷That is, $R_{NS}(t, T) = L(t) + S(t) \left(\frac{1 - \exp(-\phi\tau)}{\phi\tau} \right) + C(t) \left(\frac{1 - \exp(-\phi\tau)}{\phi\tau} - \exp(-\phi\tau) \right)$. Interestingly, the original NS article, p. 475, also notes that “This model may also be derived as an approximation to the solution in the unequal roots case by expanding in a power series in the difference between the roots.” The connection to the present article is coincidental however; from a mathematical perspective, the second-order differential equation assumed in NS happens to produce the same general solution as a bivariate first-order differential equation, which would be a minimal multifactor GATSM without an allowance for stochastic dynamics. From an economic perspective, it would be theoretically untenable to propose an N^{th} -order differential equation as the basis for a generic interest rate model.

⁸This assumes one component each to represent the near-zero and non-zero groups of eigenvalues from the generic GATSM, which is arguably the minimal model one would want for empirical work. An “over-parsimonious” NS model would be to use just a Slope component, in which case the AF version would be the Vasicek (1977) model. The absolutely most parsimonious NS model would be to use just a Level component. That is the basis for the traditional “duration” calculations from Macauley (1938) that are often used to gauge the price-sensitivity of interest rate securities to a level shift in the yield curve. The AF version would be the Vasicek (1977) model with the limit of a zero mean-reversion parameter.

⁹That is, $\exp(-\lambda_n\tau) \simeq 1 - \lambda_n\tau$, and so $\sum_{n=1}^{n_0} q_n(t) \exp(-\lambda_n\tau) \simeq \sum_{n=1}^{n_0} q_n(t) - \tau \cdot \sum_{n=1}^{n_0} q_n(t) \lambda_n$.

¹⁰While it would be tempting to interpret time variation in ϕ as representing time variation in the mean-reversion matrix K_Q , a generic GATSM that formally allowed for such flexibility would not necessarily result in factor loadings reducible to the NS form using the Taylor approximation approach as in section 2.

¹¹The theoretical case for consistency was originally proposed in Björk and Christensen (1999) and further established in Filipović (1999, 2000).

¹²I thank an anonymous referee for pointing out both the article and its implication in the context of the present article. The practical relevance of AF terms for NS models has also been questioned in Coroneo et al. (2008).

¹³All the results in this and the following subsection follow from straightforward but tedious calculus and algebra. Full workings of all the results are contained in appendix B of the working paper version of the article.

¹⁴Hamilton (1994) chapter 13 provides appropriate background to this and general aspects regarding the application of the Kalman filter. Note that the parameter ψ was latter updated using the $\beta(t)$ coefficients obtained from an initial maximum likelihood estimation, and another maximum likelihood estimation was undertaken to provide the final estimates reported subsequently. However, the additional iterations made an immaterial difference to the initial results. The estimates of $\sigma_\eta^2/(1-\psi^2)$ were around 3 percentage points, which is quite diffuse (as would be expected from a near-unit-root process).

¹⁵See the discussion in Hamilton and Wu (2010). That article and Joslin et al. (2010) have proposed techniques to alleviate econometric issues associated with latent-factor GATSMs.

¹⁶The AF/NS(2) model and its Kalman filter set-up is equivalent to the bivariate latent-factor GATSM and set-up from Babbs and Nowman (1999), but with a limit of zero mean reversion for one of the factors.

¹⁷This restriction could easily be modified if one wanted to allow for the possibility of a pair of complex conjugate eigenvalues with positive real parts. From an economic perspective, that would correspond to an expectation that innovations to the state variables would follow the product of a sinusoidal cycle and an exponential decay (rather than just an exponential decay) when returning to equilibrium.

¹⁸The estimated parameters $\sigma_v^2(\tau_k)$ are not reported here and for the EA/AF/NS(2) model in the following section to save space. The typical values were respectively 0.10 and 0.16 percentage points for the 3-m/15-y and 3-m/30-y data.

¹⁹The point estimates are 1.77 percentage points (pps) for sample A 3-m/15-y, 1.60 pps for sample B 3-m/30-y, and 1.49 bps sample C 3-m/30-y. The sample B 3-m/15-y point estimate is 2.47 pps, but the results using the 3-m/30-y data should in principle be more reliable given they exploit the additional information from longer maturities.

²⁰Besides the differences discussed, there are others that are not critical from an econometric perspective; i.e. RW assumes a lower-diagonal mean-reversion matrix for the state variables, and uses maximum-likelihood estimation assuming no measurement errors for the interest rate data of two selected maturities. The maturity span of the data is also different, from 1 month to 5 years.

²¹See, for example, Hull (2000) p. 570. The associated $A_n(t, T)$ terms have the form $-a_0\tau + A^*(\tau)$, and so have no influence on the factor loadings.

Table 1:
NS models from a generic GATSM perspective

NS model ⁽⁵⁾	Components	Representation of GATSM		AF
		$\lambda_n \simeq 0^{(1)}$	$\lambda_n \gg 0^{(2)}$	
DLY2008	2	1 [L]	1 [S]	No
HJM1992	2	1 [L]	1 [S]	Yes ⁽³⁾
Variant 1	2	1 [L]	1 [S]	Yes
NS1987	3	1 [L]	2 [S, C]	No
Krip.2006	3	1 [L]	2 [S, C]	Yes ⁽³⁾
CDR2010	3	1 [L]	2 [S, C]	Yes
Variant 2	3	1 [L]	1 [S ₁] 1 [S ₂]	Y/N
Bliss1997 ⁽⁴⁾	3	1 [L]	1 [S ₁ , ~] 1 [~, C ₂]	No
Variant 3	4	2 [L ₁ , L ₂]	2 [S ₁ , C ₁]	Y/N
Sven.1995 ⁽⁴⁾	4	1 [L]	2 [S ₁ , C ₁] 1 [~, C ₂]	No
SF2004 ⁽⁴⁾	4	1 [L]	2 [S ₁ , C ₁] 1 [~, C ₂]	Yes
CDR2009	5	1 [L]	2 [S ₁ , C ₁] 2 [S ₂ , C ₂]	Yes
Variant 4	6	2 [L ₁ , L ₂]	2 [S ₁ , C ₁] 2 [S ₂ , C ₂]	Y/N
Variant 5	6	3 [L ₁ , L ₂ , L ₃]	3 [S, C, C*]	Y/N

Notes: (1) entry is the number of terms in the Taylor expansion around 0 followed by the related NS component/s; (2) as for (1), but expansion is around $\phi = \text{mean}(\lambda_{n_0+1}, \dots, \lambda_N)$, or $\phi_1 = \text{mean}(\lambda_{n_0+1}, \dots, \lambda_{n_1})$ etc. depending on the number of groups assumed for the eigenvalues $\lambda_n \gg 0$; (3) innovations are assumed to be independent; (4) without S_2 , the model cannot be a Taylor approximation of the generic GATSM; (5) Diebold et al. (2008), Heath et al. (1992), Nelson and Siegel (1987), Krippner (2006), Christensen et al. (2009), Bliss (1997), Svensson (1995), Sharef and Filipovic (2004), Christensen et al. (2009), and NS model variants.

Table 2:
Parameter estimates for the AF/NS(2) model with 3-m/15-y data

Parameter	Sample A+B		Sample A		Sample B	
ϕ	0.4994	(0.0083)	0.7028	(0.0140)	0.3931	(0.0082)
$\gamma_{0,1}$	0.1428	(0.0029)	0.1255	(0.0033)	0.1514	(0.0046)
$\gamma_{0,2}$	0.3079	(0.0130)	0.4659	(0.0223)	0.2223	(0.0169)
σ_1	0.0225	(0.0004)	0.0241	(0.0004)	0.0206	(0.0007)
σ_2	0.0339	(0.0014)	0.0367	(0.0018)	0.0295	(0.0021)
ρ	0.5729	(0.0307)	0.6728	(0.0427)	0.5257	(0.0403)
log L	17424.7		9706.7		8803.3	
H ₀ :A=B	910.5	[0.0000]				

Note: (standard errors), [probabilities]

Table 3:

Parameter estimates for the AF/NS(2) model with 3-m/30-y data

Parameter	Sample B+C		Sample B		Sample C	
ϕ	0.3733	(0.0059)	0.3884	(0.0069)	0.3355	(0.0078)
$\gamma_{0,1}$	0.1410	(0.0026)	0.1435	(0.0031)	0.1192	(0.0043)
$\gamma_{0,2}$	0.2392	(0.0114)	0.2895	(0.0151)	0.0158	(0.0183)
σ_1	0.0172	(0.0002)	0.0172	(0.0002)	0.0191	(0.0003)
σ_2	0.0238	(0.0011)	0.0250	(0.0013)	0.0213	(0.0015)
ρ	0.3136	(0.0381)	0.4098	(0.0380)	-0.3324	(0.0621)
log L	13621.6		10292.5		5095.4	
$H_0:B=C$	3532.4	[0.0000]				

Note: (standard errors), [probabilities]

Table 4:

Estimates for EA/AF/NS(2) model with 3-m/15-y data

Parameters	Sample A+B		Sample A		Sample B	
ϕ	0.5030	(0.0085)	0.7315	(0.0142)	0.3887	(0.0083)
$\gamma_{0,1}$	0.1355	(0.0123)	0.0914	(0.0122)	0.1923	(0.0056)
$\gamma_{0,2}$	0.2985	(0.0270)	0.5194	(0.0330)	0.1459	(0.0192)
σ_1	0.0227	(0.0006)	0.0193	(0.0006)	0.0276	(0.0007)
σ_2	0.0362	(0.0022)	0.0256	(0.0023)	0.0542	(0.0026)
ρ	0.3260	(0.2336)	0.1479	(0.5765)	0.4716	(0.1822)
$\gamma_{1,11}$	0.3081	(0.4400)	0.0002	(0.0326)	1.1247	(0.7635)
$\gamma_{1,12}$	-0.2610	(0.7073)	0.1892	(0.1555)	-0.0235	(0.6909)
$\gamma_{1,21}$	-0.3353	(0.4897)	-0.1925	(0.0673)	-0.0140	(3.7293)
$\gamma_{1,22}$	0.2920	(0.4770)	0.0152	(0.2650)	0.4468	(0.8513)
log L	17429.3		9106.5		8835.7	
$H_0:A=B$	1025.7	[0.0000]				
$H_0:\gamma_1=0$	9.3	[0.0548]	59.6	[0.0000]	64.8	[0.0000]

Note: (standard errors), [probabilities]

Table 5:
 Estimates for EA/AF/NS(2) model with 3-m/30-y data

Parameters	Sample B+C		Sample B		Sample C	
ϕ	0.3319	(0.0053)	0.3904	(0.0068)	0.3177	(0.0078)
$\gamma_{0,1}$	0.1385	(0.0112)	0.1332	(0.0090)	0.1232	(0.0566)
$\gamma_{0,2}$	0.2314	(0.0307)	0.3014	(0.0266)	0.2383	(0.0593)
σ_1	0.0182	(0.0003)	0.0169	(0.0003)	0.0219	(0.0008)
σ_2	0.0236	(0.0008)	0.0240	(0.0007)	0.0217	(0.0053)
ρ	0.0317	(0.2371)	0.1695	(0.2201)	-0.9920	(1.5593)
$\gamma_{1,11}$	0.2776	(0.1245)	0.1739	(0.1823)	6.8665	(6.0884)
$\gamma_{1,12}$	-0.0516	(0.2160)	-0.0565	(0.3121)	0.8017	(1.8018)
$\gamma_{1,21}$	-0.2274	(0.1165)	-0.2182	(0.0980)	1.7737	(1.4409)
$\gamma_{1,22}$	0.0701	(0.1153)	0.1092	(0.1693)	0.2415	(0.3182)
log L	14982.3		10294.5		5180.2	
$H_0:B=C$	984.8	[0.0000]				
$H_0:\gamma_1=0$	2721.3	[0.0000]	4.0	[0.4066]	169.7	[0.0000]

Note: (standard errors), [probabilities]





