Asset pricing in a Lucas “fruit-tree” economy with non-additive beliefs*

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Abstract

We study a Lucas (1978) “fruit-tree” economy under the assumption that agents are Choquet expected utility (CEU) rather than standard expected utility (EU) decision makers. The agents’ non-additive beliefs about the economy’s stochastic dividend payment process may thus express ambiguity attitudes and accommodate violations of Savage’s sure-thing principle as elicited by Ellsberg (1961). As our main formal result we establish the existence of a unique stationary equilibrium price function for the assets in this economy. In order to account for the dynamic inconsistency of CEU decision makers, we thereby use an equilibrium concept that combines the market clearing condition of general equilibrium theory with Bayesian Nash equilibrium. A simple example about the equity premium in our economy with non-additive beliefs illustrates our formal findings.

Keywords: Non-additive Probability Measures, Choquet Expected Utility Theory, Dynamic Inconsistency, Asset Pricing, Equity Premium Puzzle

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1 Introduction

In a Lucas "fruit-tree" economy agents with identical preferences can trade on spot markets assets which are claims to future dividend payments that follow a time-homogenous Markov process. Lucas (1978) establishes existence and uniqueness of the equilibrium in this economy whereby equilibrium asset prices are characterized by a stationary price function. This seminal asset-pricing model makes the two - implicit - assumptions that, firstly, the agents are expected utility maximizers such that, secondly, their subjective beliefs correctly reflect the objective probabilities of the dividend payment process.

In this paper we study Lucas' "fruit-tree" economy under the assumption that agents are Choquet expected utility (CEU) decision makers. CEU decision makers maximize expected utility with respect to probability measures that are not necessarily additive (Schmeidler 1986,1989; Gilboa 1987). As a generalization of standard expected utility theory, CEU theory is capable of accommodating paradoxes of the Ellsberg (1961) type according to which real-life decision makers violate Savage's sure-thing principle. When restricted to the domain of gains, CEU theory is formally equivalent to *cumulative prospect theory* (Tversky and Kahneman, 1992; Wakker and Tversky, 1993) which generalizes the celebrated *prospect theory* of Kahneman and Tversky (1979).

As our main formal result we establish the existence of a unique stationary equilibrium price function and characterize its properties. By abandoning Savage's sure-thing principle CEU decision making is, firstly, dynamically inconsistent and, secondly, gives rise to a multitude of perceivable Bayesian update rules for non-additive beliefs. In order to address dynamic inconsistency our equilibrium concept takes account of the strategic situation in which different future agents play against each other. While Lucas' recursive equilibrium definition is equivalent to an Arrow-Debreu equilibrium if the commodity space includes all possible realizations of the dividend payment process, our definition of an equilibrium combines the market clearing condition of general equilibrium theory with the concept of Bayesian Nash equilibrium generalized to non-additive beliefs. Thus, in contrast to a rational expectations equilibrium approach, according to which asset prices are determined by the objective probabilities of the dividend payment process, equilibrium asset prices in our CEU economy are determined by non-additive subjective beliefs that may reflect ambiguity attitudes. The generation of such conditional non-additive beliefs for all time period agents is formally described by a time-homogenous stochastic process that satisfies the one-step-ahead Markov property. Our formal definition of a stochastic process with respect to a non-additive probability measure thereby includes the relevant Bayesian update rule in order to address the existence of different perceivable update rules in CEU decision making.

Most related to our approach are Epstein and Wang (1994) and, to a lesser extend[1], Hansen, Sargent, and Tallarini (1999) who also consider a Lucas "fruit-tree" economy under the assumption that the agents are not necessarily EU decision makers whose beliefs are given as unique additive probability measures. Motivated by the max-min expected utility (MMEU) approach of Gilboa and Schmeidler (1989), these authors consider agents who resolve their uncertainty not by a unique additive probability measure but rather by some set of additive probability measures. Instead of directly adopting MMEU preferences, however, Epstein and Wang proceed with a recursive definition of the agents' utility functional in order to ensure dynamically consistent decision making. Furthermore, Epstein and Wang simply "[...] obviate the need for an updating rule" (Epstein and Wang p. 294) by taking conditional beliefs which satisfy the Markov property as primitives of their approach. In later contributions, Epstein and Schneider (2003), as well as Hansen, Sargent, Turmuhambetova, and Williams (2006) for a continuous time framework, consider proper MMEU preferences and provide formal (*rectangularity*) conditions on priors such that the updated preferences obey dynamic consistency. While MMEU theory is closely related to CEU theory restricted to *convex* capacities, the similarity between these approaches and our decision-theoretic framework ends here. As one main difference, the restriction to dynamically consistent preferences excludes preferences that strictly violate Savage's sure-thing principle as elicited in Ellsberg paradoxes. While Epstein and Schneider (2003) regard dynamic consistency as an inadmissible principle of dynamic decision making, Hansen et al. (2006) conclude:

> "If multiple priors truly are a statement of a decision maker's subjective beliefs, we think it is not appropriate to dismiss such beliefs on the grounds of dynamic inconsistency. Repairing that inconsistency through the enlargements necessary to induce rectangularity reduces the content of the original set of prior beliefs. In our context, this enlargement is immense, too immense to be interesting to us." (p. 78)

Similarly, Eichberger, Grant, and Kelsey (2006) argue that ambiguity attitudes as elicited in Ellsberg paradoxes and inconsistencies in dynamic decision making are hardly separable and any attempt of doing so would be overtly restrictive. Since our CEU approach does not exclude dynamically inconsistent decision behavior, our model can accommodate a broader notion of ambiguity attitudes, including ambiguity attitudes

[1]An agent in the *robust control* approach of Hansen, Sargent, and Tallarini (1999) fears possible misspecification errors in his model. Thus, while not directly developed within the MMEU framework of Gilboa and Schmeidler (1989), the decision situation of the period t agents in the robust control approach can be re-interpreted as a max-min decision situation with respect to error-contaminated priors.

that are not compatible with the sure-thing principle. In contrast to Epstein and Wang (1994) and Hansen, Sargent, and Tallarini (1999), the present paper therefore attempts to incorporate rather than to exclude dynamic inconsistent decision making in order to capture relevant aspects of real-life decision making. Also in contrast to these approaches, we emphasize the important role played by the choice of the Bayesian update rule in the definition of the stochastic process that generates the agents' conditional beliefs.

The remainder of the analysis is structured as follows. Section 2 introduces CEU theory and non-additive beliefs. In Section 3 it is demonstrated that a violation of Savage's sure-thing principle implies dynamically inconsistent update rules. Three relevant examples of Bayesian update rules for non-additive beliefs are presented in Section 4. Section 5 describes stochastic processes with respect to non-additive beliefs. In Section 6 we describe the economy and introduce our equilibrium concept. We state and prove our main formal result in Section 7. In Section 8 a simple example is presented that illustrates the difference between the expected utility and the CEU approach with respect to the so-called equity premium puzzle. Finally, Section 9 concludes.

2 Choquet decision theory and non-additive beliefs

In this section we briefly recall basic elements of Choquet expected utility theory. CEU theory was first axiomatized by Schmeidler (1986, 1989) within the Anscombe and Aumann (1963) framework, which assumes preferences over objective probability distributions. Subsequently, Gilboa (1987) as well as Sarin and Wakker (1992) have presented CEU axiomizations within the Savage (1954) framework, assuming a purely subjective notion of likelihood. CEU theory is equivalent to *cumulative prospect theory* (Tversky and Kahneman 1992, Wakker and Tversky 1993) restricted to the domain of gains (compare Tversky and Wakker 1995). Moreover, as a representation of preferences over lotteries, CEU theory coincides with *rank dependent utility theory* as introduced by Quiggin (1981, 1982). Within the context of CEU theory, properties of such capacities are used in the literature for formal definitions of, e.g., *ambiguity* and *uncertainty attitudes* (Schmeidler 1989; Epstein 1999; Ghirardato and Marinacchi 2002), *pessimism* and *optimism* (Eichberger and Kelsey 1999; Wakker 2001), as well as *sensitivity to changes in likelihood* (Wakker 2004).

Let us consider a measurable space $(\Omega, \mathcal{F})$ with $\mathcal{F}$ denoting a σ-algebra on the state space Ω and a non-additive probability measure (=*capacity*) $\nu : \mathcal{F} \to [0, 1]$ satisfying

(i) $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$

(ii) $A \subset B \Rightarrow \nu(A) \leq \nu(B)$ for all $A, B \in \mathcal{F}$.

The Choquet integral of a bounded function $f : \Omega \rightarrow \mathbb{R}$ with respect to capacity ν is defined as the following Riemann integral extended to domain Ω (Schmeidler 1986):

$$E[f, \nu(d\omega)] = \int_{-\infty}^{0} (\nu(\{\omega \in \Omega \mid f(\omega) \geq z\}) - 1)\, dz + \int_{0}^{+\infty} \nu\{\omega \in \Omega \mid f(\omega) \geq z\}\, dz \tag{1}$$

whereby we will simply write $E[f, \nu]$ for $E[f, \nu(d\omega)]$. For example, in case f takes on m different values definition (1) becomes

$$E[f, \nu] = \sum_{i=1}^{m} f(\omega_i) \cdot [\nu(A_1 \cup ... \cup A_i) - \nu(A_1 \cup ... \cup A_{i-1})],$$

with $\nu(A_1 \cup A_0) = 0$ whereby $A_1, ..., A_m$ is the partition of Ω such that $f(\omega_1) > ... > f(\omega_m)$ for $\omega_i \in A_i$. While we have

$$E[f + g, \nu] = E[f, \nu] + E[g, \nu] \tag{2}$$

for functions f and g that are *comonotonic*, i.e., for all $s, t \in \Omega$,

$$(f(s) - g(s))(f(t) - g(t)) \geq 0,$$

additivity of $E[\cdot, \nu]$ does not necessarily hold for arbitrary functions f and g when ν is no additive probability measure.

For the remainder of the paper we assume that the agents have CEU preferences over Savage-acts. Recall that a Savage-act f maps the state space into some set of consequences, i.e., $f : \Omega \rightarrow X$. The Choquet expected utility of Savage-act f with respect to ν is then defined as the Choquet expected value $E[w(f), \nu]$ where $w : X \rightarrow \mathbb{R}$ denotes a vNM utility function which is unique up to a positive affine transformation.

3 Violation of the sure-thing principle and dynamic inconsistency

CEU theory has been developed in order to accommodate paradoxes of the Ellsberg type which show that real-life decision-makers violate Savage's *sure-thing principle*. Because of the abandoning of the sure-thing principle there are two important implications for conditional CEU preferences over Savage-acts. First, in contrast to Bayesian updating of additive probability measures, there exist several perceivable Bayesian update rules for non-additive probability measures (cf. Gilboa and Schmeidler 1993, Sarin and Wakker 1998, Pires 2002, Eichberger, Grant and Kelsey 2006, Siniscalchi 2001, 2006). Second, any preferences that (strictly) violate the sure-thing principle cannot be updated in a dynamically consistent way. That is, there does not exist any updating rule for capacities such that ex-ante CEU preferences that (strictly) violate the sure-thing principle are updated in a dynamically consistent manner to ex-post CEU preferences. Before we address the first implication in Section 4 let us elaborate in some detail on the second implication.

Define a Savage-act $f_B h : \Omega \to X$ such that

$$f_B h(\omega) = \begin{cases} f(\omega) & \text{for } \omega \in B \\ h(\omega) & \text{for } \omega \in \neg B \end{cases}$$

where B is some non-empty event. Recall that Savage's sure-thing principle states that, for all acts f, g, h, h' and all events $B \in \mathcal{F}$,

$$f_B h \succeq g_B h \text{ implies } f_B h' \succeq g_B h'. \tag{3}$$

Let us interpret event B as new information received by the agent. The sure-thing principle then implies a straightforward way for deriving ex-post preferences $\succeq_B$, conditional on the new information B, from the agent's original preferences $\succeq$ over Savage-acts. Namely, we have

$$f \succeq_B g \text{ if and only if } f_B h \succeq g_B h \text{ for any } h, \tag{4}$$

implying for a subjective EU decision-maker

$$f \succeq_B g \Leftrightarrow E[w(f), \mu(d\omega \mid B)] \geq E[w(g), \mu(d\omega \mid B)].$$

$E[w(f), \mu(d\omega \mid B)]$ denotes here the expected utility of act f with respect to the conditional additive probability measure $\mu(\cdot \mid B)$ defined, for all $A, B \in \mathcal{F}$ with $\mu(B) > 0$, by

$$\mu(A \mid B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

It is well known that the updating of EU preferences satisfies *dynamic consistency* which - informally - states that there are no strict ex-post incentives for deviating from an ex-ante optimal plan of actions. Formally, we define dynamic consistency in terms of update rules, i.e., rules that derive conditional preferences, $\{\succeq_B\}$ for all events B, from an ex-ante preference ordering $\succeq$.

Definition: Dynamic Consistency. *We speak of a dynamically consistent update rule iff for all ("information") partitions $\mathcal{P} \subseteq \mathcal{F}$ and all Savage-acts f, g, $f \succeq_B g$ for all $B \in \mathcal{P}$ implies $f \succeq g$.*

Observation 1. *There does not exist any dynamically consistent update rule for preferences $\succeq$ that strictly violate the sure-thing principle.*

Proof: For preferences that strictly violate the sure-thing principle we have, for some f and g,

$$f_B h \succ g_B h \text{ and } g_B h' \succ f_B h' \text{ for some } h \neq h' \text{ and some } B \in \mathcal{F}.$$

Observe that any update rule for preferences must result in conditional preferences $f \succeq_B g$ or $g \succeq_B f$. Let $\mathcal{P} = \{B, \neg B\}$ and consider at first the case $f \succeq_B g$. Since $h' \succeq_{\neg B} h'$, dynamic consistency implies $f_B h' \succeq g_B h'$, a contradiction to $g_B h' \succ f_B h'$ by the definition of a preference ordering. Now consider the case $g \succeq_B f$. Since $h \succeq_{\neg B} h$, dynamic consistency implies $g_B h \succeq f_B h$, a contradiction to $f_B h \succ g_B h$.□

4 Bayesian updating of non-additive beliefs

In case the sure-thing principle is violated, the specification of act h in (4) is no longer arbitrary so that there exist for CEU preferences several possibilities of deriving ex post preferences from ex ante preferences. That is, in the CEU framework there exist several perceivable ways of defining a conditional capacity $\nu(\cdot \mid B)$ such that

$$f \succeq_B g \Leftrightarrow E\left[w(f), \nu(\cdot \mid B)\right] \geq E\left[w(g), \nu(\cdot \mid B)\right]$$

for all B. In what follows we present three prominent update rules for capacities.

Let us at first consider conditional CEU preferences satisfying, for all acts f, g,

$$f \succeq_B g \text{ if and only if } f_B h \succeq g_B h$$

where h is the so-called conditional certainty equivalent of g, i.e., h is the constant act such that $g \sim_B h$. The corresponding Bayesian update rule for the non-additive beliefs of a CEU decision maker is the so-called full Bayesian update rule which results in the following conditional capacities (Eichberger, Grant, and Kelsey 2006)

$$\nu^{FB}(A \mid B) = \frac{\nu(A \cap B)}{\nu(A \cap B) + 1 - \nu(A \cup \neg B)} \tag{5}$$

for $A, B \in \mathcal{F}$.

In addition to the full Bayesian update rule let us also consider the class of so-called *h-Bayesian update rules* for preferences $\succeq$ over Savage acts as introduced by Gilboa and Schmeidler (1993). That is, we consider some collection of conditional preference orderings, $\left\{\succeq_B^h\right\}$ for all events B, such that for all acts f, g

$$f \succeq_B^h g \text{ if and only if } f_B h \succeq g_B h \tag{6}$$

where

$$h = \left(x^*, E; x_*, \neg E\right), \tag{7}$$

with x^* denoting the best and x_* denoting the worst consequence possible and $E \in \mathcal{F}$. For the so-called *optimistic* update rule h is the constant act where $E = \emptyset$. That is, under the optimistic update rule the null-event, $\neg B$, becomes associated with the worst consequence possible. Gilboa and Schmeidler (1993) offer the following psychological motivation for this update rule:

> "[...] when comparing two actions given a certain event B, the decision maker implicitly assumes that had B not occurred, the worst possible outcome [...] would have resulted. In other words, the behavior given B [...] exhibits 'happiness' that B has occurred; the decisions are made as if we are always in 'the best of all possible worlds'."

As corresponding optimistic Bayesian update rule for conditional beliefs of CEU decision makers we obtain

$$\nu^{opt}\left(A \mid B\right) = \frac{\nu\left(A \cap B\right)}{\nu\left(B\right)}. \tag{8}$$

For the *pessimistic* (or Dempster-Shafer) update rule h is the constant act where $E = \Omega$, associating with the null-event, $\neg B$, the best consequence possible. The psychological interpretation for this update rule according to Gilboa and Schmeidler (1993) is as follows:

> "[...] we consider a 'pessimistic' decision maker, whose choices reveal the hidden assumption that all the impossible worlds are the best conceivable ones."

The corresponding pessimistic Bayesian update rule for CEU decision makers is

$$\nu^{pess}\left(A \mid B\right) = \frac{\nu\left(A \cup \neg B\right) - \nu\left(\neg B\right)}{1 - \nu\left(\neg B\right)}. \tag{9}$$

5 Stochastic processes with non-additive beliefs

Since we are ultimately interested in uncertainty with respect to period t dividend-payments, we now impose further structure on the measure space $(\nu, \Omega, \mathcal{F})$ in order to describe a stochastic process with respect to ν. Because there does not exist a unique definition for conditional capacities we are going to include the relevant Bayesian update rule in the definition of a stochastic process.

Fix some Bayesian update rule for capacities such that $\{\succeq_B\}$ for all events B is well-defined and denote by $\nu(\cdot \mid \cdot)$ the corresponding conditional capacity. Consider a sequence of random variables $(X_t)_{t\geq 1}$ taking on values in $(X, \mathcal{X})$ where X is a (non-empty) complete separable metric space and $\mathcal{X}$ denotes the Borel σ-algebra in X. Let $X_\infty = \times_{t=1}^{\infty} X$ and consider a T*-rectangle* set $B \subseteq X_\infty$ such that

$$B = A_1 \times ... \times A_T \times X_\infty$$

where $A_t \in \mathcal{X}$ for $t = 1, ..., T$. Denote by $\mathcal{A}_T$ the collection of all T*-rectangle* sets B and define $\mathcal{F}_T$ as the σ-algebra generated by $\mathcal{A}_T$; that is, $\mathcal{F}_T$ is the intersection of all σ-algebras that contain the T*-rectangle* sets. We also refer to $\mathcal{F}_T$ as the *standard product algebra in* $X_T = \times_{t=1}^{T} X$. Similarly, denote by $\mathcal{F}_\infty$ the σ-algebra generated by $\mathcal{A}_\infty$. Obviously, $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}_\infty$ so that the $(\mathcal{F}_t)_{t\geq 1}$ constitute a *filtration*. Let $\mathcal{P}_t$ denote the finest partition of Ω contained in $\mathcal{F}_t$ and let us interpret $\mathcal{P}_t$ as the agent's "information partition" in time period t. By the filtration property, the information partitions $(\mathcal{P}_t)_{t\geq 1}$ become *finer* with increasing time, i.e., for every $P \in \mathcal{P}_{t+1}$ there is some $P' \in \mathcal{P}_t$ such that $P \subseteq P'$, and we have $\mathcal{P}_\infty = X_\infty$. This formally reflects the ideas that (i) the agent does not loose any information with the passing of time and that (ii) he will eventually know the true state of the world if he lives forever.[2] Set now $(\Omega, \mathcal{F}) = (X_\infty, \mathcal{F}_\infty)$ and observe that each random variable $X_t(\omega) = X_t(x_1, x_2, ...) = x_t$ is $\mathcal{F}_t$-measurable, which completes our construction of the stochastic process $(\nu(\cdot \mid \cdot), \Omega, \mathcal{F}) = (\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$.

For a stochastic process $(\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$ let $\nu = \nu(\cdot \mid X_\infty)$ denote the unconditional capacity on $(X_\infty, \mathcal{F}_\infty)$. The finite-dimensional capacity $\nu_{t,...,t+n}$ for $X_t \times ... \times X_{t+n}$, with $t \geq 1$ and $n \geq 0$, is then defined by

$$\begin{aligned} \nu_{t,...,t+n}(A_t \times ... \times A_{t+n}) &= \nu(X \times ... \times X \times A_t \times ... \times A_{t+n} \times X_\infty) \\ &= \nu\left(\bigcap_{s=t}^{t+n} (X \times ... \times X \times A_s \times X_\infty)\right) \end{aligned}$$

[2]An equivalent way of constructing the stochastic process $(\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$ would be to start out with a definition of the sequence of information partitions $(\mathcal{P}_t)_{t\geq 1}$ and define each $\mathcal{F}_t$ as the σ-algebra in Ω generated by $\mathcal{P}_t$.

for all $A_t \times ... \times A_{t+n} \in \mathcal{F}_n$. A stochastic process $(\nu, X_\infty, \mathcal{F}_\infty)$ is said to be *stationary* iff all finite-dimensional capacities $\nu_{t,...,t+n}$ are independent of t.

Similarly, for a given stochastic process $(\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$ define the finite-dimensional conditional capacity $\nu_{t,...,t+n}(\cdot \mid \cdot)$ for $X_t \times ... \times X_{t+n}$, with $t \geq 1$ and $n \geq 0$, given $A_s \times ... \times A_{s+m} \in \mathcal{F}_m$ as

$$\begin{aligned} & \nu_{t,...,t+n}(A_t \times ... \times A_{t+n} \mid A_s \times ... \times A_{s+m}) \\ = & \ \nu(X \times ... \times X \times A_t \times ... \times A_{t+n} \times X_\infty \mid X \times ... \times X \times A_s \times ... \times A_{s+m} \times X_\infty) \end{aligned} \tag{10}$$

for all $A_t \times ... \times A_{t+n} \in \mathcal{F}_n$. The following definition formally expresses the idea that the agent's beliefs about the immediate future $T+1$ are exclusively determined by events in period T so that the previous history, up to time period $T-1$, has no impact on these beliefs.

Definition: The One-Step-Ahead Markov Property. *We say that the stochastic process* $(\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$ *has the one-step-ahead Markov property iff, for all* T,

$$\nu_{T+1}(A_{T+1} \mid A_1 \times ... \times A_T) = \nu_{T+1}(A_{T+1} \mid A_T)$$

for any $A_t \in \mathcal{X}$ *with* $t = 1, ..., T+1$.

Definition: Time Homogeneity (=Stationary Transitions). *We say that a stochastic process* $(\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$ *is time-homogenous iff all finite-dimensional conditional capacities* $\nu_{t,...,t+n}(\cdot \mid \cdot)$ *are independent of* t.

Observe that we have for a time-homogenous process

$$\begin{aligned} & \nu(X \times ... \times X \times A_{T+1} \times X_\infty \mid X \times ... \times X \times A_T \times X_\infty) \\ = & \ \nu(X \times ... \times X \times A_{S+1} \times X_\infty \mid X \times ... \times X \times A_S \times X_\infty) \end{aligned} \tag{11}$$

whenever $A_T = A_S$ and $A_{T+1} = A_{S+1}$. In the case of time-homogeneity we will therefore slightly abuse notation and write

$$\nu(A_{T+1} \mid A_T)$$

for (11) or $\nu_{T+1}(A_{T+1} \mid A_T)$.

Finally, we will need a technical definition ensuring that the conditional Choquet expected value of a bounded real-valued continuous function in X is itself a bounded real-valued continuous function in X.

Definition: The Feller Property. *Consider a time-homogenous stochastic process* $(\nu(\cdot \mid \cdot), X_{\infty}, \mathcal{F}_{\infty})$ *with the one-step-ahead Markov property. We say that* $\nu(\cdot \mid \cdot)$ *satisfies the "Feller property" iff for every bounded continuous function* $f : X \rightarrow \mathbb{R}$, *the function* $Tf : X \rightarrow \mathbb{R}$ *defined by*

$$(Tf)(x) = E[f, \nu(dx' \mid x)]$$

is also bounded and continuous, i.e., if $f \in C[X]$ *then* $Tf \in C[X]$ *whereby* $C[X]$ *denotes the space of bounded real-valued continuous functions on* X *endowed with the supremum norm* $\|\cdot\|_{\infty}$.

Remark. To see that the inclusion of the update rule in the definition of a stochastic process is relevant for non-additive beliefs assume for a moment that ν reduces to an additive probability measure, say μ. An important class of (additive) stationary stochastic processes that also trivially satisfy the one-step-ahead Markov property as well as time-homogeneity are i.i.d. processes $(\mu(\cdot \mid \cdot), X_{\infty}, \mathcal{F}_{\infty})$ such that $\mu(A_{T+1} \mid A_T) = \mu(A_{T+1})$ for all A_{T+1}, A_T. If we want to define i.i.d. processes with respect to a non-additive measure ν our concept of independence will depend on the chosen update rule. For example, we would say that A_{T+1} is independent of A_T with respect to $\nu(\cdot \mid \cdot)$ iff ν satisfies

for full Bayesian updating

$$\nu(A) = \frac{\nu(A \cap B)}{\nu(A \cap B) + 1 - \nu(A \cup \neg B)},$$

for optimistic updating

$$\nu(A) = \frac{\nu(A \cap B)}{\nu(B)},$$

and for pessimistic updating

$$\nu(A) = \frac{\nu(A \cup \neg B) - \nu(\neg B)}{1 - \nu(\neg B)},$$

whereby

$$\begin{aligned} A &: = X \times \ldots \times X \times A_{T+1} \times X_{\infty} \in \mathcal{F}_{\infty} \\ B &: = X \times \ldots \times X \times A_T \times X_{\infty} \in \mathcal{F}_{\infty}. \end{aligned}$$

6 The economy

Consider the decision-situation of a representative period t agent who has initial endowment $\mathbf{z}_t = (z_{1,t}, \ldots, z_{k,t}) \in [0,1]^k$ of k different assets whereby $\mathbf{z}_0 = \mathbf{1}$. The agent has to choose the amounts of assets he is going to buy, respectively sell, on the period t spot market; that is, he effectively decides about his period $t+1$ asset holdings $\mathbf{z}_{t+1} \in [0,1]^k$. For realized dividend payments $\mathbf{y}_s = (y_{1,s}, ..., y_{k,s}) \in \mathbb{R}^k$ per asset unit and ex-dividend spot market asset prices $\mathbf{p}_s = (p_{1,s}, ..., p_{k,s}) \in \mathbb{R}^k$, the period $s = t, t+1, ...$ consumption from the agent's perspective is given as follows

$$c(\mathbf{y}_s, \mathbf{p}_s, \mathbf{z}_s, \mathbf{z}_{s+1}) = \mathbf{y}_s \cdot \mathbf{z}_s + \mathbf{p}_s \cdot (\mathbf{z}_s - \mathbf{z}_{s+1}).$$

In our model the period t agents are Choquet decision makers who are uncertain about future dividend-payments. In order to describe this uncertainty in terms of a stochastic process we consider a sequence of random variables $(\mathbf{Y}_t)_{t \geq 1}$ defined on Ω which take on values in $(\mathbf{Y}, \mathcal{Y})$ where $\mathbf{Y}$ is a non-empty compact subset of the Euclidean space $\mathbb{R}^k$ and $\mathcal{Y}$ is the Borel σ-algebra in $\mathbf{Y}$. Denote by $\mathcal{F}_T$ the standard product algebra in $\mathbf{Y}_T = \times_{t=1}^{T} \mathbf{Y}$ and observe that the $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}_\infty$ constitute a filtration where $\mathcal{F}_\infty$ denotes the standard product algebra in $\mathbf{Y}_\infty = \times_{t=1}^{\infty} \mathbf{Y}$. Finally, define $\mathbf{Y}_t(\omega) = \mathbf{Y}_t(\mathbf{y}_1, \mathbf{y}_2, ...) = \mathbf{y}_t$ whereby we interpret $\mathbf{y}_t = (y_{1,t}, ..., y_{k,t})$ as the realized period t dividend-payments in the state of the world ω. Analogously to (10), a period t agent's belief about future dividend payments $A_{t+1} \times ... \times A_{t+1+n} \in \mathcal{F}_n$ under the condition that he has observed the history $\mathbf{y}_1, ..., \mathbf{y}_t \in \times_{i=1}^{t} \mathbf{Y}$ of dividend payments is defined as the finite-dimensional conditional capacity $\nu_{t+1,...,t+1+n}(\cdot \mid \cdot)$ such that

$$\begin{aligned} & \nu_{t+1,...,t+1+n}(A_{t+1} \times ... \times A_{t+1+n} \mid \mathbf{y}_1, ..., \mathbf{y}_t) \\ = \; & \nu(\mathbf{Y} \times ... \times \mathbf{Y} \times A_{t+1} \times ... \times A_{t+1+n} \times \mathbf{Y}_\infty \mid \{\mathbf{y}_1\} \times ... \times \{\mathbf{y}_t\} \times \mathbf{Y}_\infty). \end{aligned}$$

For a given non-additive belief ν defined on $(\Omega, \mathcal{F})$ and a fixed Bayesian update rule we have thus constructed a stochastic process $(\nu(\cdot \mid \cdot), \Omega, \mathcal{F}) = (\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$, which formally describes how the period t agents' conditional non-additive beliefs about the economy's dividend-payments are generated.

Assumptions on beliefs. *We assume that $(\nu(\cdot \mid \cdot), X_\infty, \mathcal{F}_\infty)$ is a time-homogenous stochastic process with the one-step-ahead Markov property such that $\nu(\cdot \mid \cdot)$ satisfies the Feller property. In particular, we assume:*

(A1) *The "One-step-ahead Markov property": For all $t \geq 0$ and all histories $\mathbf{y}_1, ..., \mathbf{y}_t \in \times_{i=1}^{t} \mathbf{Y}$,*

$$\nu_{t+1}(A_{t+1} \mid \mathbf{y}_1, ..., \mathbf{y}_t) = \nu_{t+1}(A_{t+1} \mid \mathbf{y}_t)$$

for all $A_{t+1} \in \mathcal{Y}$.

(A2) *"Time-homogeneity": For all $t \geq 0$ and all $\mathbf{y}_t \in \mathbf{Y}$,*

$$\nu_{t+1}\left(A_{t+1} \mid \mathbf{y}_t\right) = \nu\left(A_{t+1} \mid \mathbf{y}_t\right)$$

for all $A_{t+1} \in \mathcal{Y}$.

(A3) *The "Feller property": Let $f : \mathbf{Y} \to \mathbb{R}$ be any real-valued continuous function in $\mathbf{Y}$ and define $Tf : \mathbf{Y} \to \mathbb{R}$ such that*

$$(Tf)(\mathbf{y}) = E\left[f, \nu\left(d\mathbf{y}' \mid \mathbf{y}\right)\right].$$

Then Tf is also a real-valued continuous function in $\mathbf{Y}$.

Contingent on the realized state of the world $(\mathbf{y}_1, ..., \mathbf{y}_t, \mathbf{y}_{t+1}, ...) \in \Omega$, a period t agent gains vNM utility

$$w\left(\mathbf{z}_{t+1}\right) = \sum_{s=t}^{\infty} \beta^{s-t} u\left(c\left(\mathbf{y}_s, \mathbf{p}_s, \mathbf{z}_s, \mathbf{z}_{s+1}\right)\right)$$

from choosing (conditional Savage act) $\mathbf{z}_{t+1}$ for given $\mathbf{z}_s$, $s \neq t+1$, and $\mathbf{p}_s$, $s = 0, 1, ...$, whereby $\beta \in (0,1)$ is the time-discount factor, $u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, strictly increasing, strictly concave and bounded. Let $\mathbf{y}_\infty$ denote the generic element of $\mathbf{Y}_\infty$. Contingent on the observed history of dividend payments $(\mathbf{y}_1, ..., \mathbf{y}_t) \in \times_{j=1}^{t} \mathbf{Y}$, the Choquet expected utility of period t agent's asset holding choice is then given as[3]

$$\begin{aligned} & E\left[w\left(\mathbf{z}_{t+1}\right), \nu\left(d\mathbf{y}_\infty \mid \mathbf{y}_1, ..., \mathbf{y}_t\right)\right] \\ = \; & u\left(c\left(\mathbf{y}_t, \mathbf{p}_t, \mathbf{z}_t, \mathbf{z}_{t+1}\right)\right) + E\left[\sum_{s=t+1}^{\infty} \beta^{s-t} u\left(c\left(\mathbf{y}_s, \mathbf{p}_s, \mathbf{z}_s, \mathbf{z}_{s+1}\right)\right), \nu\left(d\mathbf{y}_\infty \mid \mathbf{y}_1, ..., \mathbf{y}_t\right)\right]. \end{aligned} \tag{12}$$

Let us, somewhat informally, describe this economy as a Bayesian game such that the period $t = 1, 2, ...$ agents are different players whose beliefs about "nature's moves" are given as conditional capacities. A strategy of a period t agent is then any $\mathcal{F}_t$-measurable function $\mathbf{f}_{t+1} : \times_{j=1}^{t} \mathbf{Y} \to [0,1]^k$; that is, a player's strategy maps the set of possible histories into the set of possible asset holdings at period $t+1$. Since consumption in time periods $s > t+1$ is irrelevant to the first-order conditions of the maximization problem (12), we obtain the following characterization of an equilibrium under the assumptions that, firstly, all period t agents choose mutually best responses with respect to belief $\nu(\cdot \mid \cdot)$ and, secondly, at equilibrium prices demand equals supply in each time period.

[3] Notice that this definition of the Choquet expected utility of an uncertain infinite consumption stream coincides for the multiple-prior framework with definition (2.4.3) discussed by Epstein and Wang (1994).

Definition: *The strategy profile* $\left\{\mathbf{f}_{t+1}^*\right\}_{t=1,2...}$ *and the price-sequence* $\{\mathbf{p}_t^*\}_{t=1,2,...}$ *constitute an equilibrium of this economy if and only if, for all histories* $(\mathbf{y}_1, ..., \mathbf{y}_t) \in \times_{j=1}^t \mathbf{Y}$,

$$\begin{aligned} \mathbf{f}_{t+1}^*(\mathbf{y}_1, ..., \mathbf{y}_t) &= \arg \max_{\mathbf{z}_{t+1} \in [0,1]^k} u\left(c\left(\mathbf{y}_t, \mathbf{p}_t^*, \mathbf{f}_t^*, \mathbf{z}_{t+1}\right)\right) \\ &\quad + \beta E\left[u\left(c\left(\mathbf{y}_{t+1}, \mathbf{p}_{t+1}^*, \mathbf{z}_{t+1}, \mathbf{f}_{t+2}^*\right)\right), \nu\left(d\mathbf{y}_{t+1} \mid \mathbf{y}_t\right)\right] \end{aligned} \tag{13}$$

such that $\mathbf{p}_t^*$ *and* $\mathbf{p}_{t+1}^*$ *ensure*

$$\mathbf{z}_{t+1}^* = \mathbf{z}_t^* \tag{14}$$

for all t.

Because of the market-clearing condition (14) we know that in any equilibrium every period t agent must optimally hold the initial endowment of assets; that is, $\mathbf{f}_t^* = \mathbf{1}$ for all t. In order to study the equilibria of this economy, it therefore remains to characterize the (shadow) prices which support this allocation. The assumptions on beliefs allow us to consider a stationary situation in which every period t agent, $t = 0, 1, ...$, faces the same maximization problem. As a consequence, we can restrict attention to stationary equilibrium price functions.

Observation 2. *A stationary price function* $\mathbf{p}^* : \mathbf{Y} \to \times_{j=1}^k \mathbb{R}_+$ *is an equilibrium price-function if and only if, for all* $\mathbf{y}_t, \mathbf{y}_{t+1} \in \mathbf{Y}$,

$$\begin{aligned} \mathbf{f}_{t+1}^* &= \mathbf{1} = \arg \max_{\mathbf{z}_{t+1} \in [0,1]^k} u\left(c_t\left(\mathbf{y}_t, \mathbf{p}^*(\mathbf{y}_t), \mathbf{f}_t^* = \mathbf{1}, \mathbf{z}_{t+1}\right)\right) \\ &\quad + \beta E\left[u\left(c_{t+1}\left(\mathbf{y}_{t+1}, \mathbf{p}^*(\mathbf{y}_{t+1}), \mathbf{z}_{t+1}, \mathbf{f}_{t+2}^* = \mathbf{1}\right)\right), \nu\left(d\mathbf{y}_{t+1} \mid \mathbf{y}_t\right)\right] \end{aligned}$$

for all t.

7 The main result

In this section we state and prove our main formal result which characterizes the asset-pricing equilibrium in the economy. Our formal proof is thereby based on a contraction mapping argument that is similar to Lucas' formal argument in the case of expected utility decision makers with additive beliefs.

Proposition. *There exists a unique continuous equilibrium price-function* $\mathbf{p}^* : \mathbf{Y} \to \times_{j=1}^k \mathbb{R}_+$ *such that, for* $j = 1, ..., k$,

$$p_j^*(\mathbf{y}) = \frac{f_j^*(\mathbf{y})}{u'(\mathbf{y} \cdot \mathbf{1})}, \; \mathbf{y} \in \mathbf{Y}$$

whereby $f_j^* \in C[\mathbf{Y}]$ is the unique fixed point of the operator $T : C[\mathbf{Y}] \to C[\mathbf{Y}]$ defined by

$$(Tf_j)(\mathbf{y}) = \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot y_j' + f_j(\mathbf{y}'), \nu(d\mathbf{y}' \mid \mathbf{y})\right], \mathbf{y} \in \mathbf{Y}.$$

Moreover, as approximation for the fixed-point f_j^* we have for any $f_j \in C[\mathbf{Y}]$

$$\left\|T^n f_j - f_j^*\right\|_\infty \leq \beta^n \left\|f_j - f_j^*\right\|_\infty.$$

Proof: The corresponding FOC's of the maximization problem are

$$p_{j,t}^*(\mathbf{y}_t) \cdot u'(\mathbf{y}_t \cdot \mathbf{1}) = \beta E\left[u'(\mathbf{y}_{t+1} \cdot \mathbf{1}) \cdot \left(y_{j,t+1} + p_{j,t+1}^*(\mathbf{y}_{t+1})\right), \nu(d\mathbf{y}_{t+1} \mid \mathbf{y}_t)\right]$$

for $j = 1, ..., k$. Since the problem is stationary, we have $\mathbf{p}_t^*(\mathbf{y}) = \mathbf{p}^*(\mathbf{y})$ for all $\mathbf{y}$ and t so that the equilibrium price function is, after dropping the time indices, characterized by

$$p_j^*(\mathbf{y}) \cdot u'(\mathbf{y} \cdot \mathbf{1}) = \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot \left(y_j' + p_j^*(\mathbf{y}')\right), \nu(d\mathbf{y}' \mid \mathbf{y})\right]$$

for $j = 1, ..., k$. Define now the operator T such that

$$(Tf_j)(\mathbf{y}) = \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot y_j' + f_j(\mathbf{y}'), \nu(d\mathbf{y}' \mid \mathbf{y})\right]$$

and observe that $p_j^* : \mathbf{Y} \to \mathbb{R}$ exists and is unique if and only if T has a unique fixed point $f_j^* = \left(Tf_j^*\right)$ so that $p_j^* = \frac{f_j^*}{u'}$. Since $\mathbf{Y}$ is compact, any real valued continuous function f in $\mathbf{Y}$ is also bounded so that $f \in C[\mathbf{Y}]$ whereby $C[\mathbf{Y}]$ denotes the space of bounded real-valued continuous functions in $\mathbf{Y}$ endowed with the supremum norm $\|\cdot\|_\infty$. Recall that $C[\mathbf{Y}]$ is a complete metric space. By the Feller property, the function $Tf_j : \mathbf{Y} \to \mathbb{R}$ is also continuous so that the operator T maps the complete metric space $C[\mathbf{Y}]$ into itself. As a consequence, we can apply the contraction mapping theorem according to which there exists a unique fixed point of T if there exists some number (=modulus) $c < 1$ such that, for all functions $f, g \in C[\mathbf{Y}]$,

$$\|Tf - Tg\|_\infty \leq c \cdot \|f - g\|_\infty.$$

By Theorem 5 in Blackwell (1965), if T is monotone and satisfies, for all functions f_j and any constant a,

$$[T(f_j + a)](\mathbf{y}) \leq (Tf_j)(\mathbf{y}) + c \cdot a \tag{15}$$

for some $c < 1$, then T is a contraction with modulus c. Since the Choquet integral is monotone, so is T. In order to prove that T is a contraction, it therefore remains to be shown that condition (15) is satisfied for some $c < 1$. Observe that a constant function a is comonotonic to any function, so that, by (2),

$$\begin{aligned} & \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot y_j' + f_j(\mathbf{y}') + a, \nu(d\mathbf{y}' \mid \mathbf{y})\right] \\ \leq \; & \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot y_j' + f_j(\mathbf{y}'), \nu(d\mathbf{y}' \mid \mathbf{y})\right] + c \cdot a \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} & \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot y_j' + f_j(\mathbf{y}'), \nu(d\mathbf{y}' \mid \mathbf{y})\right] + \beta E\left[a, \nu(d\mathbf{y}' \mid \mathbf{y})\right] \\ \leq \; & \beta E\left[u'(\mathbf{y}' \cdot \mathbf{1}) \cdot y_j' + f_j(\mathbf{y}'), \nu(d\mathbf{y}' \mid \mathbf{y})\right] + c \cdot a \end{aligned}$$

$$\Leftrightarrow$$

$$\beta \leq c.$$

Thus, set $c = \beta$ to see that (15) is satisfied for some $c < 1$.□

Since $E[a \cdot f, \nu] = aE[f, \nu]$ for any constant a, we immediately obtain from the first order conditions the following CEU counterpart of a familiar result from EU asset pricing theory.

Corollary. *Expressed in terms of the gross-return of asset j, i.e.,*

$$R_j^* = \frac{p_j^*(\mathbf{y}') + y_j'}{p_j^*(\mathbf{y})}$$

and the so-called stochastic discount factor

$$M = \beta \cdot \frac{u'(\mathbf{y}' \cdot \mathbf{1})}{u'(\mathbf{y} \cdot \mathbf{1})},$$

the economy's equilibrium is characterized by the following conditions

$$1 = E\left[M \cdot R_j^*, \nu(d\mathbf{y}' \mid \mathbf{y})\right] \tag{16}$$

for $j = 1, ..., k$.

8 An illustrative example: The equity premium

Based on the Lucas fruit-tree economy, Mehra and Prescott (1985) study a model with a stationary productivity *growth rate* process for which they observe the so-called "equity premium puzzle". According to this puzzle, a realistically calibrated model implies a much lower difference in gross returns between a "risky" and a "risk-free" asset than observed in the empirical data. Although there is no productivity growth in our fruit-tree economy, it is nevertheless instructive to have a look at the formal relationship between asset returns in our Choquet economy and compare the case between additive and non-additive beliefs. In order to focus the analysis, we will restrict attention to a sub-class of non-additive probability measures defined as *neo-additive capacities.*

8.1 Neo-additive capacities

The concept of *neo-additive capacities* has been introduced by Chateauneuf, Eichberger, and Grant (2007).

Definition. *For a given measurable space $(\Omega, \mathcal{F})$ the neo-additive capacity, ν, is defined, for some $\delta, \lambda \in [0,1]$ by*

$$\nu(A) = \delta \cdot (\lambda \cdot \omega^o(A) + (1-\lambda) \cdot \omega^p(A)) + (1-\delta) \cdot \pi(A) \tag{17}$$

for all $A \in \mathcal{F}$ such that π is some additive probability measure and we have for the non-additive capacities ω^o

$$\begin{aligned} \omega^o(A) &= 1 \text{ if } A \neq \emptyset \\ \omega^o(A) &= 0 \text{ if } A = \emptyset \end{aligned}$$

and ω^p respectively

$$\begin{aligned} \omega^p(A) &= 0 \text{ if } A \neq \Omega \\ \omega^p(A) &= 1 \text{ if } A = \Omega. \end{aligned}$$

The parameter δ can be interpreted as the decision makers *degree of ambiguity* whereas λ stands for his *degree of optimism.* The motivation for these interpretations is immediate from the following observation, which extends a result (Lemma 3.1) of Chateauneuf, Eichberger, and Grant (2007) from the case of finite random variables to the more general case of random variables with a closed and bounded range.

Observation 3. *Let f be real-valued function with closed and bounded range. Then the Choquet expected value (1) of f with respect to a neo-additive capacity (17) is given by*

$$E[f, \nu] = \delta(\lambda \max f + (1-\lambda) \min f) + (1-\delta) E[f, \pi]. \tag{18}$$

Proof: Relegated to the appendix.

Observation 4. *Suppose the non-additive belief of the agent is given as the neo-additive capacity (17). If the agent applies the full Bayesian update rule, his conditional non-additive belief is given by*

$$\nu^{FB}(A \mid B) = \delta_B \cdot \lambda + (1-\delta_B) \cdot \pi(A \mid B), \tag{19}$$

for $A, B \in \mathcal{F}$, *whereby*

$$\delta_B = \frac{\delta}{\delta + (1-\delta) \cdot \pi(B)}. \tag{20}$$

Proof: Relegated to the appendix.

8.2 The equity premium

Under the assumption that the agents have neo-additive beliefs and apply the full Bayesian update rule, the economy's equilibrium conditions (16) are given as

$$1 = \delta_{\mathbf{y}} \left(\lambda \max\left(M \cdot R_j^*\right) + (1-\lambda) \min\left(M \cdot R_j^*\right)\right) + (1-\delta_{\mathbf{y}}) E\left[M \cdot R_j^*, \pi(d\mathbf{y}' \mid \mathbf{y})\right],$$

or equivalently,

$$\begin{aligned} 1 \;=\; & \delta_{\mathbf{y}} \left(\lambda \max\left(M \cdot R_j^*\right) + (1-\lambda) \min\left(M \cdot R_j^*\right)\right) \\ & + (1-\delta_{\mathbf{y}}) \left(Cov\left[M, R_j^*, \pi(d\mathbf{y}' \mid \mathbf{y})\right] + E\left[M, \pi(d\mathbf{y}' \mid \mathbf{y})\right] \cdot E\left[R_j^*, \pi(d\mathbf{y}' \mid \mathbf{y})\right]\right) \end{aligned}$$

for $j = 1, .., k$, whereby

$$\delta_{\mathbf{y}} = \frac{\delta}{\delta + (1-\delta) \cdot \pi(\{\mathbf{y}\} \times \mathbf{Y}_\infty)}.$$

For a "risk-free" asset we have, by definition, constant equilibrium returns R_f^* implying

$$1 = R_f^* \left(\delta_{\mathbf{y}} (\lambda \max M + (1-\lambda) \min M) + (1-\delta_{\mathbf{y}}) E\left[M, \pi(d\mathbf{y}' \mid \mathbf{y})\right]\right).$$

Let

$$\rho = \delta_{\mathbf{y}} (\lambda \max M + (1-\lambda) \min M) + (1-\delta_{\mathbf{y}}) E\left[M, \pi^*\right]$$

and observe that the following relationship holds between equilibrium returns of a risky, R_j^*, and a risk-free, R_f^*, asset in this economy

$$\begin{aligned} R_f^* &= \frac{\delta_{\mathbf{y}}}{\rho}\left(\lambda \max\left(M \cdot R_j^*\right) + (1-\lambda)\min\left(M \cdot R_j^*\right)\right) \\ &+ \frac{(1-\delta_{\mathbf{y}})}{\rho} Cov\left[M, R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right] \\ &+ \frac{(1-\delta_{\mathbf{y}})\, E\left[M, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]}{\rho} E\left[R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]. \end{aligned} \tag{21}$$

In case the neo-additive capacities reduce to additive beliefs, i.e., $\delta_{\mathbf{y}} = 0$, we have $\rho = E\left[M, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]$ so that we obtain the following familiar characterization of the equilibrium equity premium in terms of gross-returns

$$\begin{aligned} R_f^* &= \frac{Cov\left[M, R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]}{E\left[M, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]} + E\left[R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right] \Leftrightarrow \\ E\left[R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right] - R_f^* &= \frac{-Cov\left[M, R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]}{E\left[M, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]}. \end{aligned}$$

Standard formulations of the equity-premium puzzle proceed by assuming that the subjective additive belief $\pi\left(d\mathbf{y}' \mid \mathbf{y}\right)$ coincides with the "objective" probability of the economy's dividend-payments, say π^*. According to the equity premium puzzle the empirically observed value of the equity premium

$$E\left[R_j^*, \pi^*\right] - R_f^* \tag{22}$$

is then much higher as the value of

$$\frac{-Cov\left[M, R_j^*, \pi^*\right]}{E\left[M, \pi^*\right]} \tag{23}$$

if the model parameters (e.g., the risk-aversion coefficient in the case of a CRRA utility function) are calibrated with realistic values.[4] While any thorough discussion of the equity premium puzzle is beyond the scope of this paper, it is immediately obvious from (21) that the formal equivalence between the empirically observed equity premium (22) and (23) does no longer hold in the case of non-additive beliefs. The theoretical relationship between $E\left[R_j^*, \pi\left(d\mathbf{y}' \mid \mathbf{y}\right)\right]$ and R_f^* is much more complex in this CEU economy with neo-additive beliefs than under the assumption of expected utility maximizers because it additionally depends on the agent's degree of ambiguity, $\delta_{\mathbf{y}}$, and his degree of

[4] For reviews on the extremely rich literature on the equity premium- and related asset return puzzles see the survey articles by Kocherlakota (1996), Campbell (2003), Mehra and Prescott (2003) and the textbook treatments in Cochrane (2001) and Duffie (2001).

optimism, λ, in a non-trivial way. For example, in the case of extreme ambiguity, i.e., $\delta_{\mathbf{y}} = 1$, we have

$$R_f^* = \frac{\lambda \max\left(M \cdot R_j^*\right) + (1-\lambda)\min\left(M \cdot R_j^*\right)}{\lambda \max M + (1-\lambda)\min M}$$

so that the equilibrium return of the risk-free asset would be completely independent of any ("objective") expected return, $E\left[R_j^*, \pi^*\right]$, of the risky asset since the agent only cares about the worst, resp. best, possible return of the asset. That is, for $\delta_{\mathbf{y}} = 1$ the model's implications are compatible with any empirically observed equity premium (22). While this case describes an admittedly extreme scenario of CEU decision-making, it illustrates that the standard equity premium formula (23) is apparently not very robust with respect to a generalization from additive to non-additive beliefs.

Finally observe that even in case we restrict attention to the comparably simple class of neo-additive capacities the economy's equilibrium conditions (16) depend on the applied update rule. That is, the specific relationship (21) between the equilibrium returns of a risk-free and a risky asset is only valid under the assumption that the agents apply the full Bayesian update rule. If we used instead, e.g., the optimistic or the pessimistic update rule, different equilibrium conditions would obtain.

9 Concluding remarks

We study a Lucas "fruit-tree" economy under the assumption that the agents are Choquet decision makers who have non-additive beliefs about the economy's dividend-payments. As main formal result we establish conditions such that there exists a unique asset-pricing equilibrium in this economy. Our equilibrium concept thereby takes account of the fact that Choquet decision makers are dynamically inconsistent whenever they violate Savage's sure-thing principle. In particular, our equilibrium concept combines the market clearing condition of general equilibrium theory with the concept of Bayesian Nash equilibrium with respect to non-additive beliefs. The conditional non-additive beliefs of all time period agents are thereby generated by a time-homogenous stochastic process that satisfies the one-step-ahead Markov property and that explicitly states the Bayesian update rule applied by the agents.

We present a simple example which illustrates that the assumption of non-additive beliefs may result in a theoretical relationship between the equilibrium returns of assets which strongly differs from the standard results obtained under the assumption of additive beliefs. The introduction of CEU preferences to consumption based asset pricing models opens, in our opinion, an interesting avenue for future research with respect to asset return puzzles such as, e.g., the equity premium puzzle. This is especially true since the CEU approach is founded on behavioral axioms that have proved useful in the description of real-life decision making in experimental situations.

Appendix

Proof of observation 3: By an argument in Schmeidler (1986), it suffices to restrict attention to a non-negative valued random variable f so that

$$E[f,\nu] = \int_0^{+\infty} \nu\{\omega \in \Omega \mid f(\omega) \geq z\}\, dz, \tag{24}$$

which is equivalent to

$$E[f,\nu] = \int_{\min f}^{\max f} \nu\{\omega \in \Omega \mid f(\omega) \geq z\}\, dz$$

since the range of f is closed and bounded. We consider a partition P_n, $n = 1, 2, ...$, of Ω with members

$$A_n^k = \{\omega \in \Omega \mid a_{k,n} < f(\omega) \leq b_{k,n}\} \text{ for } k = 1, ..., 2^n$$

such that

$$\begin{aligned} a_{k,n} &= [\max f - \min f] \cdot \frac{(k-1)}{2^n} + \min f \\ b_{k,n} &= [\max f - \min f] \cdot \frac{k}{2^n} + \min f. \end{aligned}$$

Define the step functions $a_n : \Omega \to \mathbb{R}$ and $b_n : \Omega \to \mathbb{R}$ such that, for $\omega \in A_n^k$, $k = 1, ..., 2^n$,

$$\begin{aligned} a_n(\omega) &= a_{k,n} \\ b_n(\omega) &= b_{k,n}. \end{aligned}$$

Obviously,

$$E[a_n,\nu] \leq E[f,\nu] \leq E[b_n,\nu]$$

for all n and

$$\lim_{n\to\infty} E[b_n,\nu] - E[a_n,\nu] = 0.$$

That is, $E[a_n,\nu]$ and $E[b_n,\nu]$ converge to $E[f,\nu]$ for $n \to \infty$. Furthermore, observe that

$$\begin{aligned} \min a_n &= \min f \text{ for all } n\text{, and} \\ \max b_n &= \max f \text{ for all } n. \end{aligned}$$

Since $\lim_{n\to\infty} \min b_n = \lim_{n\to\infty} \min a_n$ and $E[b_n,\pi]$ is continuous in n, we have

$$\begin{aligned} \lim_{n\to\infty} E[b_n,\nu] &= \delta\left(\lambda \lim_{n\to\infty} \max b_n + (1-\lambda) \lim_{n\to\infty} \min b_n\right) + (1-\delta) \lim_{n\to\infty} E[b_n,\pi] \\ &= \delta(\lambda \max f + (1-\lambda)\min f) + (1-\delta) E[f,\pi]. \end{aligned}$$

In order to prove proposition 3, it therefore remains to be shown that, for all n,

$$E[b_n,\nu] = \delta(\lambda \max b_n + (1-\lambda)\min b_n) + (1-\delta) E[b_n,\pi].$$

Since b_n is a step function, (24) becomes

$$\begin{aligned} E[b_n,\nu] &= \sum_{A_n^k \in P_n} \nu\left(A_n^{2^n} \cup ... \cup A_n^k\right) \cdot (a_{k,n} - a_{k-1,n}) \\ &= \sum_{A_n^k \in P_n} a_{k,n} \cdot \left[\nu\left(A_n^{2^n} \cup ... \cup A_n^k\right) - \nu\left(A_n^{2^n} \cup ... \cup A_n^{k-1}\right)\right], \end{aligned}$$

implying for a neo-additive capacity

$$\begin{aligned} E[b_n,\nu] &= \max b_n \left[\delta\lambda + (1-\delta)\pi\left(A_n^{2^n}\right)\right] + \sum_{k=2}^{2^n-1} a_{k,n}(1-\delta)\pi\left(A_n^k\right) \\ &\quad + \min b_n \left[1 - \delta\lambda - (1-\delta)\sum_{k=2}^{2^n} \pi\left(A_n^k\right)\right] \\ &= \delta\lambda \max b_n + (1-\delta)\sum_{k=1}^{2^n} a_{k,n}\pi\left(A_n^k\right) + \min b_n [\delta - \delta\lambda] \\ &= \delta(\lambda \max b_n + (1-\lambda)\min b_n) + (1-\delta) E[b_n,\pi]. \end{aligned}$$

□

Proof of observation 4: An application of the full Bayesian update rule to a neo-additive capacity gives

$$\begin{aligned} \nu^{FB}(A \mid B) &= \frac{\delta \cdot \lambda + (1-\delta)\cdot\pi(A\cap B)}{\delta\cdot\lambda + (1-\delta)\cdot\pi(A\cap B) + 1 - (\delta\cdot\lambda + (1-\delta)\cdot\pi(A\cup\neg B))} \\ &= \frac{\delta\cdot\lambda + (1-\delta)\cdot\pi(A\cap B)}{1 + (1-\delta)\cdot(\pi(A\cap B) - \pi(A\cup\neg B))} \\ &= \frac{\delta\cdot\lambda + (1-\delta)\cdot\pi(A\cap B)}{1 + (1-\delta)\cdot(\pi(A\cap B) - \pi(A) - \pi(\neg B) + \pi(A\cap\neg B))} \\ &= \frac{\delta\cdot\lambda + (1-\delta)\cdot\pi(A\cap B)}{1 + (1-\delta)\cdot(-\pi(\neg B))} \\ &= \frac{\delta\cdot\lambda + (1-\delta)\cdot\pi(A\cap B)}{\delta + (1-\delta)\cdot\pi(B)} \\ &= \frac{\delta\cdot\lambda}{\delta + (1-\delta)\cdot\pi(B)} + \frac{(1-\delta)\cdot\pi(B)}{\delta + (1-\delta)\cdot\pi(B)}\pi(A\mid B) \\ &= \delta_B\cdot\lambda + (1-\delta_B)\cdot\pi(A\mid B) \end{aligned}$$

with δ_B given by (20).□

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