The Stability of Walrasian General Equilibium

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Abstract

We prove the stability of equilibrium in a completely decentralized Walrasian general equilibrium economy in which prices are fully controlled by economic agents, with production and trade occurring out of equilibrium.

Journal of Economic Literature Classifications:
C62—Existence and Stability Conditions of Equilibrium
D51—Exchange and Production Economies
D58—Computable and Other Applied General Equilibrium Economies

1 Introduction

Walras (1954 [1874]) developed a general model of competitive market exchange, but provided only an informal argument for the existence of a market-clearing equilibrium for this model. Wald (1951 [1936]) provided a proof of existence for a simplified version of Walras' model, and this proof was substantially generalized by Debreu (1952), Arrow and Debreu (1954), Gale (1955), Nikaido (1956), McKenzie (1959), Negishi (1960), and others.

The stability of the Walrasian economy was a central research focus in the years following the existence proofs (Arrow and Hurwicz 1958, 1959, 1960; Arrow, Block and Hurwicz 1959; Nikaido 1959; McKenzie 1960; Nikaido and Uzawa 1960). Following Walras' tâtonnement process, these models assumed that there is no production or trade until equilibrium prices are attained, and out of equilibrium, there is a price vector shared by all agents, the time rate of change of which is a function of excess demand. These efforts at proving stability were successful only by assuming narrow and implausible conditions (Fisher 1983). Indeed,

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Scarf (1960) provided simple examples of unstable Walrasian equilibria under a tâtonnment dynamic.

Several researchers then explored the possibility that allowing trading out of equilibrium could sharpen stability theorems (Uzawa 1959, 1961, 1962; Negishi 1961; Hahn 1962; Hahn and Negishi 1962), but these effort enjoyed only limited success. Moreover, Sonnenschein (1973), Mantel (1974, 1976), and Debreu (1974) showed that any continuous function, homogeneous of degree zero in prices, and satisfying Walras' Law, is the excess demand function for some Walrasian economy. These results showed that no general stability theorem could be obtained based on the tâtonnement process. Indeed, subsequent analysis showed that chaos in price movements is the generic case for the tâtonnement adjustment processes (Saari 1985, Bala and Majumdar 1992).

A novel approach to the dynamics of large-scale social systems, evolutionary game theory, was initiated by Maynard Smith and Price (1973), and adapted to dynamical systems theory in subsequent years (Taylor and Jonker 1978, Friedman 1991, Weibull 1995). The application of these models to economics involved the shift from biological reproduction to behavioral imitation as the criterion for the replication of successful agents.

We apply this framework by treating the Walrasian economy as the stage game of an evolutionary process. We assume each agent is endowed in each period with a good that he must trade to obtain the various goods he consumes. There are no inter-period exchanges. An agent's trade strategy consists of a set of *private prices* for the good he produces and the goods he consumes, such that, according to the individual's private prices, a trade is acceptable if the value of goods received is at least as great as the value of the goods offered in exchange. The exchange process of the economy is hence defined as a multipopulaton game with private prices as strategies. We assume that the strategies of successful traders are occasionally copied by less successful traders with the same production good. With rather mild assumptions, the stability of equilibrium is then guaranteed.

2 The Walrasian Economy

We consider an economy with a finite number of goods, $h = 1, \ldots, l$, and a finite number of agents $i = 1, \ldots, m$. Agent i has $\mathbf{R}^l_+$ as consumption set, a utility function $u_i : \mathbf{R}^l_+ \to \mathbf{R}_+$ and an initial endowment $e_i \in \mathbf{R}^l_+$. This economy is denoted by $\mathcal{E}(u, e)$.

In this setting an allocation $\overline{x} \in (\mathbf{R}^l_+)^m$ of goods is *feasible* if it belongs to the set

$$\mathcal{A}(e) = \left\{ x \in (\mathbf{R}^l_+)^m \middle| \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \right\}.$$

The demand of an agent i is the mapping d_i: $\mathbf{R}^l_+ \to \mathbf{R}^l_+$ that associates to a price $p \in \mathbf{R}^l_+$ the utility-maximizing individual allocations satisfying the budget constraint. That is,

$$d_i(p) := \text{argmax}\{u(x_i) \big| x_i \in \mathbf{R}^l_+, p \cdot x_i \leq p \cdot e_i\}.$$

A feasible allocation $\overline{x} \in \mathcal{A}(e)$ then is an *equilibrium allocation* if there exists a price $\overline{p} \in S$ such that for all i, $\overline{x}_i \in d_i(\overline{p})$. We denote the set of such equilibrium prices by $\mathbf{E}(u, e)$.

To ensure that the economy satisfies conditions for the existence of such an equilibrium allocation, we first define a *quasi-equilibrium*, which consists of an attainable allocation $x^* \in \mathcal{A}(e)$ and a price $p^* \in \mathbf{R}^l_+$ such that $u_i(x_i) > u_i(x_i^*)$ implies $p^* \cdot x_i > p^* \cdot x_i^*$. To guarantee the existence of such a quasi-equilibrium, it suffices to assume

Assumption 1 (Utility) *For all $i = 1, \ldots, m$, u_i is continuous, strictly concave, monotonic and non-satiated, the latter being defined as: for all $x_i \in \mathbf{R}^l_+$, there exists $x_i' \in \mathbf{R}^l_+$ such that $u_i(x_i) > u_i(x_i')$.*

To ensure that every quasi-equilibrium is an equilibrium allocation, it suffices to assume that at a quasi-equilibrium the agents do not receive the minimal possible income (Hammond 1993, Florenzano 2005).

Assumption 2 (Income) *For every quasi-equilibrium (p^*, x^*), and for every $i = 1, \ldots, m$, there exists $x_i \in \mathbf{R}^l_+$ such that $p^* \cdot x_i^* > p^* \cdot x_i$.*

The strict concavity assumption in (1) is not necessary for the existence of a quasi-equilibrium but implies demand mappings are single-valued, what will prove useful below.

3 Exchange Processes with Private Prices

We consider exchange processes in which agents determine their behavior on the basis of private prices that represent their subjective priors concerning utility-maximizing exchange ratios for their interactions with other agents in the economy. We consider a game $\mathcal{G}(u, e, \xi)$ such that:

- Each agent has the strategy set P which is a finite subset of $\mathbf{R}^l_{++}$ of the form $P = K^l$ where $K \subset \mathbf{R}^l_+$ is a finite set of commodity prices with minimum $p_{\min} > 0$ and maximum $p_{\max} > p_{\min}$.

- There is an exchange mechanism $\xi\colon P^m \to \mathcal{A}(e)$ that associates to a profile of private prices $\pi = (p_1, \ldots, p_m)$, where p_i is the price vector of agent i, an attainable allocation $\xi(\pi) = (\xi_1(\pi), \ldots, \xi_m(\pi)) \in \mathcal{A}(e)$. We consider $\xi(\pi)$ to be the outcome of the exchange process.[1]

- The payoff to player i given strategy profile π is $v_i(\pi) = u_i(\xi_i(\pi))$.

In the following, we investigate the relationships between equilibrium allocations of the economy $\mathcal{E}(u, e)$ and Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ when additional restrictions are placed on the exchange mechanism ξ.

4 Strict Equilibria and Dynamic Stability

Given an n-players game with strategy sets $\{S_i | i = 1, \ldots, n\}$ and payoff functions $\{v_i | i = 1, \ldots, n\}$,

Definition 1 *A strategy profile* $\{s_i^* | i = 1, \ldots, n\}$ *is a* strict Nash equilibrium *if, for all i and for every* $s_i \in S_i$, $v_i(s_i^*, s_{-i}^*) > v_i(s_i, s_{-i}^*)$.

We then have[2] (Weibull 1995):

Proposition 1 *A strict Nash equilibrium of a multipopulation game is asymptotically stable for all weakly payoff-positive selection dynamics, including the replicator dynamic. Every asymptotically stable equilibrium allocation in the replicator dynamic of a multipopulation game is a strict Nash equilibrium of the game.*

We will now characterize exchange mechanisms ξ, such that for every economy $\mathcal{E}(u, e)$ the strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ correspond to the equilibrium allocations of the economy $\mathcal{E}(u, e)$.

5 A Stable Tâtonnement Process

We begin with the simple case of in which the tâtonement process is stable. In this process there is no trade unless prices are such that all markets clear. The corresponding exchange mechanism ξ satisfies the following assumption:

Assumption 3 (Tâtonement) *We call* ξ *a* tâtonnement *exchange process if*

$$\xi_i(\pi) = \begin{cases} d_i(\overline{p}) & \textit{if for all } i,\ \pi_i = \overline{p} \in \mathbf{E}(u, e) \\ e_i & \textit{otherwise.} \end{cases} \tag{1}$$

[1] We could as well consider that the game contains a chance move by nature and that the outcome is a distribution of attainable allocations. Provided each element of this distribution satisfies the assumption put forward on ξ, the same arguments apply.

[2] See also the appendix.

We say that at equilibrium allocation there are *gains from trade* if:

Assumption 4 (Gains from trade) *For every $\overline{p} \in \mathbf{E}(u, e)$, we have for all i, $u_i(d_i(\overline{p})) > u_i(e_i))$.*

Assuming a tâtonnement exchange process in which there are gains from trade at all equilibrium allocations, it is clear that:

- At a strategy profile in which two or more agents use a non-equilibrium price, each agent is allocated his initial endowment and no unilateral deviation can modify this allocation, so that the profile is a non-strict Nash equilibrium

- At a strategy profile in which all agents but one use the same equilibrium price, each agent is allocated his initial endowment. Under assumption (4), the "non-equilibrium" agent makes everyone strictly better-of by deviating to the equilibrium price as each agent is then allocated his equilibrium demand. So, such a strategy profile is not a Nash equilibrium.

- At a strategy profile in which all agents use the same equilibrium price $\overline{p}$, each agent receives the equilibrium allocation $u_i(d_i(\overline{p}))$. Under assumption (4), an agent deviating to a different price makes everyone strictly worst-of as each agent is then allocated his initial endowment. Hence, the strategy profile is a strict Nash equilibrium.

It follows that the equilibrium allocations of the economy are the only strict Nash equilibria of the game:

Proposition 2 *Under Assumption (1) and (4), $\pi \in P^m$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if and only if there exists $\overline{p} \in \mathbf{E}(u, e)$ such that for all i, $\pi_i = \overline{p}$.*

Following Proposition 1, this yields the following dynamic stability result for the equilibrium allocation of the underlying economy:

Proposition 3 *Under Assumption (1) and (4), the only asymptotically stable strategy profiles for the replicator dynamic in the game $\mathcal{G}(u, e, \xi)$ are those in which each agent uses an equilibrium price $\overline{p} \in \mathcal{E}(u, e)$ and agent i is allocated its payoff $d_i(\overline{p})$ in the equilibrium allocation.*

This stable version of the tâtonnement process is an interesting curiosity, but of little use as agents have no incentive to agree to exchanges that will in fact not be implemented. The more fundamental issue for us is the evolutionary stability of Walrasian equilibrium in a setting where agents produce and trade out of equilibrium.

6 Stability with Trading Out of Equilibrium

We now suppose that for each agent i there is one good h_i such that i is endowed some quantity of h_i at the start of each trading in each period.

Assumption 5 (Production) *For every $i = 1, \ldots, m$, there exists $h_i = 1, \ldots, l$ such that $e_{i,h} > 0 \Leftrightarrow h = h_i$.*

We refer to h_i as agent i's *production good* and call the corresponding price π_{i,h_i} the *production price* of agent i. We assume that agents either do not derive utility from the consumption of their production good, or that they bring to market only the excess over their desired consumption of their production good. With respect to the other goods we shall assume that an agent consumes a fixed subset of goods, which all are necessary. This assumption is more general than the standard one in the literature which requires that all goods are consumed in strictly positive quantities. For a subset $C \subset \{1, \cdots, l\}$, we define the vector subspace V_C as $V_C := \{x \in \mathbf{R}^l \mid x_h > 0 \Leftrightarrow h \in C\}$ and pr_{V_C} as the projection on V_C.

Assumption 6 (Desirability) *For every $i = 1, \ldots, m$, there exists $C_i \subset \{1, \cdots, l\}$ and $v_i : V_{C_i} \to \mathbf{R}_+$ such that for all $x \in \mathbf{R}^l_+$, $u_i(x) = v_i(pr_{V_{C_i}}(x))$.*

The standard assumption in the general equilibrium literature is to consider that the excess demand satisfies a boundary condition according to which demand for any good tends to infinity as its price approaches zero (Balasco 2009). As we have a finite price space, we adopt a milder condition stating that when the price of a good is minimal there necessarily is excess demand for that good:

Assumption 7 (Aggregate Nonsatiation) *For all $h = 1, \ldots, l$ and for all $p \in P$, we have: $p_h = p_{\min} \Rightarrow \sum_{i=1}^m d_{i,h}(p) > \sum_{i=1}^m e_{i,h}$.*

Finally, we adopt a technical assumption asserting that among the agents considered, there are always two replicates of a common (implicit) archetype. That is:

Assumption 8 (Replicates) *For every $i = 1, \ldots, m$, there is a unique $i' \in \{1, \ldots, m\}$ such that $u_i = u_{i'}$ and $e_i = e_{i'}$. Agents i and i' are called* replicates.

This last assumption is required in order to be able to represent some form of competition among sellers despite the fact that only one representative of each type is considered a player in the exchange encounter. It can be interpreted as requiring that the economy is a 2-fold replica of some underlying simple economy.

7 Characterizing Walrasian Exchange Mechanisms

We shall from this point onward take the assumptions of the preceding section as satisfied. We present plausible restrictions on the exchange mechanism ξ that guarantee that the only strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$ are equilibrium allocations of the economy $\mathcal{E}(u, e)$. Note that according to the Production (5) and Desirability (6) assumptions, the relevant prices for agent i are limited to that of his production good h_i and those of the set of goods he consumes C_i. The price strategy of agent i could be reduced to this subset of indexes but in order to avoid cumbersome notations, we let each agent use P as strategy space and assume that for any $h \notin C_i \cup \{h_i\}$, the agent simply uses the price of an arbitrarily chosen agent whose production good is h.

This investigation is meaningful only if there is indeed a counterpart to Walrasian equilibrium in the exchange process. The minimal assumption in this respect is that when all agents use the same equilibrium price as private price, the exchange process yields the corresponding equilibrium allocation. That is:

Assumption 9 (Equilibrium) *If $\pi \in P^m$ is such that for every $i = 1, \ldots, m$, $\pi_i = \overline{p} \in \mathbf{E}(u, e)$, we have for every $i = 1, \ldots, m$, $\xi_i(\overline{p}) = d_i(\overline{p})$.*

The essential role of private prices in the exchange mechanism is to indicate which trades are worth undertaking. We assume an agent purchases a good only if it is sold at a price not greater than his private price:

Assumption 10 (Compatibility) *For all $i = 1, \ldots, m$ and for all $h \neq h_i$, we have $\xi_{i,h}(p) > 0$ only if there exists j such that $h_j = h$ and $p_{i,h_j} \geq p_{j,h_j}$.*

We shall then call a price profile $\pi = (p_1, \ldots, p_m)$ *compatible* if it is such that for all $i = 1, \ldots, m$ for all $h \in C_i$ there exists j such that $h_j = h$ and $p_{i,h_j} \geq p_{j,h_j}$. We shall denote by $\mathcal{C} \subset P^m$ the set of compatible price profiles.

We now turn to the price and income sensitivity of the exchange mechanism. We adopt two mild monotonicity conditions. The first is that a uniform decrease of consumption prices increases the utility of an agent:

Assumption 11 (Demand Monotonicity) *If $\pi = (p_i, p_{-i})$ and $\pi' = (p'_i, p_{-i})$ are compatible price profiles such that*[3] *$p_{i,h_i} = p'_{i,h_i}$ and for all $h \in C_i$, $p'_{i,h} \leq p_{i,h}$ with at least one inequality being strict, then $u_i(\xi_i(\pi')) > u_i(\xi_i(\pi))$.*

Second, we assume that an increase of an agent's income, all other things being equal, is beneficial. Note that under Assumptions (5) and (6), utility producing income is this spent on goods other than the production good and can hence be

[3]If one wants to think in terms of normalized prices, one should read $p'_{i,h_i} \geq p_{i,h_i}$.

measured as the amount spent on goods other than the production good. For agent i at price p_i this is $p_{i,-h_i} \cdot \xi_{i,-h_i}(p)$. The income monotonicity assumption can then be stated as:

Assumption 12 (Income monotonicity) *If $\pi = (p_i, p_{-i})$ and $\pi' = (p'_i, p_{-i})$ are compatible price profiles such that*[4] *for all $j \neq i$, $p_{i,j} = p'_{i,j}$, and $p'_{i,-h_i} \cdot \xi_{i,-h_i}(p') > p_{i,-h_i} \cdot \xi_{i,-h_i}(p)$, then $u_i(\xi_i(\pi')) > u_i(\xi_i(\pi))$.*

Finally, we shall assume there exists some form of competition among sellers so that in case of excess supply, demand is allocated first to the seller with the lowest offer price and when all prices are equal, the demand is shared equally by sellers:

Assumption 13 (Competition) *Let i, j be replicates, let $h = h_i = h_j$, define $c_h = \text{card}\{i | h_i = h\}$ so average demand is given by*

$$\overline{d}_{j,h} = \frac{1}{c_h} \sum_{j=1}^{n} d_{j,h}(p).$$

- *Suppose that for all $i, i' \in \{1, \ldots, m\}$ such that $h_i = h$, we have $p_{i,h} = p_{i',h}$. Then $p_{i,-h} \cdot \xi_{i,-h}(p) = p_{i,h} \min(e_{i,h}, \overline{d}_{j,h})$.*
- *If i is such that $h_i = h$ and for all $i' \neq i$ with $h_{i'} = h$, we have $p_{i,h} < p_{i',h}$ then $p_{i,-h} \cdot \xi_{i,-h}(p) = p_{i,h} \min(\sum_{j=1}^{n} d_{j,h}(p), e_{i,h})$.*

Assumptions (9) to (13) imply that the only strict Nash equilibria of $\mathcal{G}(u, e, \xi)$ are the equilibrium allocations of $\mathcal{E}(u, e)$. Namely:

Proposition 4 *Under Assumptions (9) – (13), $\pi \in P^m$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if and only if there exists $\overline{p} \in \mathbf{E}(u, e)$ such that for all $i = 1, \ldots, m$, $\pi_i = \overline{p}$.*

Proof: Suppose π is such that for all $i = 1, \ldots, m$, we have $\pi_i = \overline{p}$, where $\overline{p} \in \mathbf{E}(u, e)$. According to Equilibrium Assumption (9), the payoff to player i, given the strategy profile π, is $u_i(d_i(\overline{p})) > 0$. Assume agent i deviates to private price p'. If there exists $h \in C_i$ such that $p'_h < \overline{p}_h$, then given that all the other agents still use price $\overline{p}$, we have according to Compatibility Assumption (10) that $\xi_{i,h}(p', \pi_{-i}) = 0$ and hence $u_i(\xi_i(p', \pi_{-i})) = 0$, and agent i is strictly worse off. Otherwise, one has for all $h \in C_i$, $p'_h \geq \overline{p}_h$ with at least one inequality being strict, given that $p' \neq \overline{p}$. Demand Monotonicity Assumption (11) then implies that

[4] If one wants to think in terms of normalized prices, one should read for all $j \neq i$, $p_{i,j} \leq p'_{i,j}$.

$u_i(\xi_i(p', \pi_{-i})) < u_i(\xi_i(\overline{p}, \pi_{-i}))$, so the deviating agent is strictly worse off. This proves that π is a strict Nash equilibrium of the game $\mathcal{G}(u, e, \xi)$.

For the other direction, we prove the contrapositive: if $\overline{p}$ is not an equilibrium allocation, then the corresponding $\pi \in P^m$ is not a strict Nash equilibrium. There are three cases.

First consider the case where there exists $\tilde{p} \notin \mathbf{E}(u, e)$ such that for all i, $\pi_i = \tilde{p}$. Using Walras' law, one has $\tilde{p} \cdot \sum_{i=1}^{n}(d_i(\tilde{p}) - e_i) = 0$. As $\tilde{p}$ is not an equilibrium price and $\tilde{p} \in \mathbf{R}^l_{++}$ we must have for some h, $\sum_{i=1}^{h} d_{i,h}(\tilde{p}) < \sum_{i=1}^{h} e_{i,h}$. Consider then an agent $i \in \{1, \cdots, m\}$ such that $h_i = h$. This agent initially has, according to Competition Assumption (13), an income $\tilde{p}_{i,-h} \cdot \xi_{i,-h}(\tilde{p}) = \overline{d}_{j,h}$. Assume this agent deviates to price p' such that for all $h' \neq h$, $p'_{h'} = \tilde{p}_{h'}$ and $p'_h = \tilde{p}_{h-}$ where $\tilde{p}_{h-}$ is the largest price less than $\tilde{p}_h$ in K. Note that K is defined in section (3) and that $\tilde{p}_h$ cannot be the minimal price in K given the Aggregate Nonsatiation Assumption (7). According to Competition Assumption (13), the income of agent i then becomes $p'_{i,-h} \cdot \xi_{i,-h}(p', \pi_{-i}) = p'_h \min(e_h, \sum_{i=1}^{m} d_{i,h}(\tilde{p}))$. Provided the price grid K is sufficiently thin [5], we then have $\tilde{p}_i \cdot \xi_i(\pi) < p' \cdot \xi_i(p', \pi_{-i})$, so that using the Income Monotonicity Assumption 12, we have $u_i(\xi_i(\pi)) < u_i(\xi_i(p', \pi_{-i}))$. This implies π is not a strict Nash equilibrium of the game $\mathcal{G}(u, e, \xi)$.

Finally, consider the case where π is such that there exists $i, j \in \{1, \ldots, m\}$ with $\pi_i \neq \pi_j$ or equivalently that there exists i, j with $\pi_{i,h_j} \neq \pi_{j,h_j}$.

- A first case is this where the price profile is not compatible. That is for at least one $i \in \{1, \ldots, m\}$ and one $h \in C_i$, one has for all j with $h_j = h$, $\pi_{i,h} < \pi_{j,h}$. The Compatibility Assumption (10) then yields that $\xi_{i,h_j}(\pi) = 0$ and $u_i(\xi_i(\pi)) = 0$, so that π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$.

- A second case is this where the price profile is compatible and one agent has a private price for a good greater than that of each potential seller, who all use the same price. That is there is $i \in \{1, \ldots, m\}$ and $h \in C_i$ such that for all j, j' with $h_j = h_{j'} = h$, $\pi_{i,h} \geq \pi_{j,h} = \pi_{j',h}$, with one of those inequalities being strict. Let then i and h be such that for all j with $h_j = h$, $\pi_{i,h} > \pi_{j,h}$. Suppose agent i deviates to p' such that $p'_h = \pi_{j,h}$ and for all $h' \neq h$, $p'_{h'} = \pi_{i,h}$. It is a direct consequence of the Demand Monotonicity Assumption (11) that $u_i(\xi_i(\pi)) \leq u_i(\xi_i(p', \pi_{-i}))$ so that π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$.

- The final case is this where the price profile is compatible and two sellers use a different price. That is $\pi \in \mathcal{C}$ and there exists $h' \in \{1, \ldots, l\}$ and $k, \ell \in$

[5] See Appendix B on the construction of the price grid

$\{1,\ldots,m\}$ such that $h_k = h_\ell = h'$ and $\pi_{k,h'} \neq \pi_{\ell,h'}$. By the Competition Assumption (13), it is always the case that one of these agents can raise his income by deviating to the price of the other: the higher price agent shall deviate if excess supply is large enough, the lower price one otherwise. The Income Monotonicity Assumption (12) then implies the deviating agent is better-of, so that π cannot be a strict Nash equilibrium of $\mathcal{G}(u,e,\xi)$.

As in the case of Proposition (1), this yields the following dynamic stability result for the equilibrium allocation of the underlying economy:

Proposition 5 *The only asymptotically stable strategy profiles for the replicator dynamic in $\mathcal{G}(u,e,\xi)$ are those for which each agent uses an equilibrium price $\overline{p} \in \mathcal{E}(u,e)$ and agent i is allocated his equilibrium allocation $d_i(\overline{p})$.*

8 Conclusion

Our proof can be summarized as follows. The equilibrium of a Walrasian market system is a strict Nash equilibrium of an exchange game in which the requirements of the exchange process are quite mild and easily satisfied. Assuming producers update their private price vectors periodically by adopting the strategies of more successful peers leads to a multipopulation game in which strict Nash equilibria are asymptotically stable in the replicator dynamic. Conversely, all stable equilibria of the replicator dynamic are strict Nash equilibria of the exchange process.

The major innovation of our model is the use of private prices, one set for each agent, in place of the standard assumption of a uniform public price faced by all agents. The traditional public price assumption would not have been auspicious even had a plausible stability theorem been available using such prices. This is because there is no mechanism for prices to change in a system of public prices—no agent can alter the price schedules faced by the large number of agents with whom any one agent has virtually no contact.

The private price assumption is the only plausible assumption for a fully decentralized market system not in equilibrium, because there is in fact no natural way to define a common price system except in equilibrium. With private prices, each individual is free to alter his price vector at will, market conditions alone ensuring that something approximating a uniform system of prices will prevail in the long run.

There are many general equilibrium models with private prices in the literature, based for the most part on strategic market games (Shapley and Shubik 1977, Sahi and Yao 1989, Giraud 2003) in which equilibrium prices are set on a market-by-market basis to equate supply and demand, and it is shown that under appropriate conditions the Nash equilibria of the model are Walrasian equilibria. These are equilibrium models, however, without known dynamical properties, and unlike our approach they depend on an extra-market mechanism to balance demand and supply.

The equations of our dynamical system are too many and too complex to solve analytically or to estimate numerically. However, it is possible to construct a discrete version of the system as a finite Markov process. The link between stochastic Markov process models and deterministic replicator dynamics is well documented in the literature. Helbing (1996) shows, in a fairly general setting, that mean-field approximations of stochastic population processes based on imitation and mutation lead to the replicator dynamic. Moreover, Benaim and Weibull (2003) show that large population Markov process implementations of the stage game have approximately the same behavior as the deterministic dynamical system implementations based on the replicator dynamic. This allows us to study the behavior of the dynamical market economy for particular parameter values.For sufficiently large population size, the discrete Markov process captures the dynamics of the Walrasian economy extremely well with near certainty (Benaim and Weibull 2003). While analytical solutions for the discrete system exist (Kemeny and Snell 1960, Gintis 2009), they also cannot be practically implemented. However, the dynamics of the Markov process model can be studied for various parameter values by computer simulation (Gintis 2007, 2012).

Macroeconomic models have been especially handicapped by the lack of a general stability model for competitive exchange. The proof of stability of course does not shed light on the fragility of equilibrium in the sense of its susceptibility to exogenous shocks and its reactions to endogenous stochasticity. These issues can be studied directly through Markov process simulations, and may allow future macroeconomists to develop analytical microfoundations for the control of excessive market volatility.

Appendix A: Asymptotic stability and replicator dynamics

Let G be an n-player game with finite strategy sets $\{S_i|i = 1, \ldots, n\}$, the cardinal of which is denoted by $k_i = |S_i|$, with strategies indexed by $h = 1, \ldots, k_i$ and payoff functions $\{\pi_i|i = 1, \ldots, n\}$. Let $\Delta_i = \{\sigma_i \in \mathbf{R}^{k_i} | \forall h, \sigma_{i,h} \geq 0$ and $\sum_{h=1}^{k_i} \sigma_{i,h} = 1\}$ the which is the mixed strategy space of agent i, and let $\Delta = \prod_{i=1}^{n} \Delta_i$. In an evolutionary game setting, an element $\sigma_i \in \Delta_i$ represents a population of players i with a share $\sigma_{i,h}$ of the population playing strategy $h \in S_i$.

Dynamics for such population of players $(\sigma_1, \ldots, \sigma_N) \in \Delta$ are defined by specifying, a growth rate function $g : \Delta \to \mathbf{R}^{\sum_{i=1}^{n} k_i}$, for all $i = 1, \ldots, n$ and

$h = 1, \ldots, k_i$:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} g_{i,h}(\sigma) \quad (2)$$

We shall restrict attention to growth-rate functions that satisfy a regularity condition and maps Δ into itself (Weibull 1995).

Definition 2 *A regular growth-rate function is a Lipschitz continuous function g defined in a neighborhood of Δ such that for all $\sigma \in \Delta$ and all $i = 1, \ldots, n$ we have has $g_i(\sigma) \cdot \sigma_i \neq 0$.*

The dynamics of interest in a game-theoretic setting are those that satisfy minimal properties of monotonicity with respect to payoffs. Strategies of player i in $B_i(\sigma) := \{s \in S_i | u_i(s, \sigma_{-i}) > u_i(\sigma)\}$ that have above average payoffs against σ_{-i}, have a positive growth-rate in the following sense:

Definition 3 *A regular growth-rate function g is weakly payoff-positive if for all $\sigma \in \Delta$ and $i = 1, \ldots, n$,*

$$B_i(\sigma) \neq 0 \Rightarrow g_{i,h} > 0 \text{ for some } s_{i,h} \in B_i(\sigma), \quad (3)$$

where $s_{i,h}$ denotes the h^{th} pure strategy of player i.

Among the class of weakly-payoff positive dynamics, the replicator dynamic is by far the most commonly used to represent the interplay between population dynamics and strategic interactions. It corresponds to the system of differential defined for all $i = 1, \ldots, n$ and $h = 1, \ldots, |S_i|$ by:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h}(\pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)). \quad (4)$$

That is thus the system of differential equation corresponding to the growth rate function $g_{i,h}(\sigma) = \pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)$.

It is standard to show that the system of differential equations (2) associated with a regular and weakly-payoff monotonic growth function has a unique solution defined at all times for every initial condition in Δ. We will generically denote the solution mapping by $\psi : \mathbf{R}_+ \times \Delta \to \Delta$, so $\psi(t, \sigma_0)$ gives the value at time t of the solution to (2) with initial condition $\sigma(0) = \sigma_0$. Stability properties of (2), are then defined in terms of this solution mapping:

Definition 4 *A strategy profile $\sigma^* \in \Delta$ is called Lyapunov stable if every neighborhood V of σ^* contains a neighborhood W of σ^* such that $\psi(t, \sigma) \in V$ for all $\sigma \in W \cap \Delta$.*

Definition 5 *A strategy profile $\sigma^* \in \Delta$ is called asymptotically stable if it is Lyapunov stable and there exists a neighborhood V of σ^* such that for all $\sigma \in V \cap \Delta$:*

$$\lim_{t \to +\infty} \psi(t, \sigma) = \sigma^*.$$

Appendix B: Construction of the price grid

In the proof of proposition (4), we require that the price grid K be sufficiently thin as to have $\tilde{p}_h \dfrac{\sum_{i=1}^m d_{i,h}(\tilde{p})}{\text{card}\{i|h_i = h\}} < p_h^- \min(e_h, \sum_{i=1}^m d_{i,h}(\tilde{p}))$

If $e_h > \sum_{i=1}^m d_{i,h}(\tilde{p})$, the condition amounts to $p_h^- > \dfrac{\tilde{p}_h}{\text{card}\{i|h_i = h\}}$, which is satisfied provided the mesh (minimum distance between two prices) in K is smaller than p_{min}.

If $\sum_{i=1}^m d_{i,h}(\tilde{p}) > e_h$, the condition amounts to $\tilde{p}_h \sum_{i=1}^m d_{i,h}(\tilde{p}) < p_h^- \sum_{i=1}^m e_h$. If one let $s_h = \sum_{i=1}^m e_h$ denote the supply, $z_h^+(\tilde{p}) = \max(\sum_{i=1}^m e_{i,h} - d_{i,h}(\tilde{p}), 0)$ the excess supply and $\alpha_h = \tilde{p}_h - p_h^-$ the price change, the condition can be rewritten as $\alpha_h < \dfrac{\tilde{p}_h z_h^+(\tilde{p}_h)}{s_h}$. In order to ensure the grid K is always thin enough for this condition to hold, it suffices to construct it as follows. Consider an arbitrary small $\epsilon > 0$, and let $B^c(p, \epsilon)$ denote the complement of the open ball with center $p \in \mathbf{R}^l$ and radius ϵ. We let $P_0 := [p_{min}, p_{max}]^l \cap_{p \in \mathbf{E}(u,e)} B^c(p, \epsilon)$. P_0 is a compact subset of $\mathbf{R}^l$ such that $\overline{\alpha} > 0$ where $\overline{\alpha} := \min_{p \in P_0} \dfrac{p_h z_h^+(p_h)}{s_h}$. In order to guarantee the desired condition hold, it suffice to choose the price mesh smaller than $\overline{\alpha}$ (but in the neighborhood of equilibrium prices).

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