Continuous Mixed Strategy Equilibria
in Static and Dynamic Tournaments

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Introduction

- In this project I study several static and dynamic games that do not have Nash equilibria in pure strategies.
- I characterize the symmetric equilibria in which the players use continuous mixed strategies.
- The main novelty in this paper is the analysis of a setting with state dynamics.
  - I derive a closed-form solution for the Markovian mixed equilibrium strategies.
  - It enables us to make predictions about players’ behavior in the short run and in the long run.
- This analysis can be used to study the bonus structure of salesmen within a company.
- Another application: to construct prizes for sport events.
Probably the most famous game with a continuous mixed strategy equilibrium is Varian’s model of sales.

- Two firms play a Bertrand game.
- Firms sell to both captive customers (who do not migrate) and opportunistic customers (who buy at the lowest price).
- There are no pure strategy equilibria:
  - each producer wants to undercut the opponent’s price in order to take advantage of the opportunistic buyers;
  - however, charging low prices will cause firms to forfeit profits from the captive customers.

There is little work on mixed strategies in dynamic games.

I am aware of two exceptions (neither exhibits state dynamics).

- War of attrition games: waiting is costly, but the player who waits the longest gets the prize.
- Preemption games: if one player chooses to act he gets the prize, but if multiple players act the prize gets destroyed.
Setup of the Stage Game

- First we look at a one-shot game of effort choice.
- There are two players: 1 and 2.
- They choose their effort levels, $x_1$ and $x_2$, simultaneously.
- Each of them is required to exert effort at least $a > 0$.
- If $x_i > x_j$, player $i$ wins.
- Both players get positive prizes!
  - $i$ gets the first prize, $\alpha P$.
  - $j$ gets the second prize, $(1 - \alpha)P$.
  - Assume that $1 > \alpha > 0.5$.

- The prize fund is increasing in the joint effort of the players:
  $$P = \beta(x_1 + x_2), \quad \text{where } 2 > \beta > 0.$$

- To recap, if $x_i > x_j$, then the payoffs of $i$ and $j$ are
  $$\alpha \beta(x_1 + x_2) - x_i, \quad (1 - \alpha)\beta(x_1 + x_2) - x_j.$$
No Nash Equilibrium in Pure Strategies

- This game does not have an equilibrium in pure strategies.
- If \( j \) exerts the minimum effort \( x_j = a \), player \( i \) will want to work slightly harder than \( j \) (i.e. choose \( x_i = a + \varepsilon \)).
- Obviously, \( j \) will then trump \( i \)'s effort by choosing \( x_j = x_i + \varepsilon \), etc.
- But if \( i \)'s effort is above some critical \( y \), \( j \) may settle for the 2nd prize.
  - if \( x_i > y \), \( j \) would be better off free-riding on \( i \)'s effort.
  - he will choose the minimum effort \( a \), and get \( (1 - \alpha)\beta(x_i + a) - a \).
- The critical value \( y \) solves \( \alpha \beta(2y) - y = (1 - \alpha)\beta(y + a) - a \), or

\[
y = \frac{[1 - \beta + \alpha \beta]a}{1 + \beta - 3\alpha \beta}.
\]

- If \( 1 + \beta - 3\alpha \beta > 0 \), then we get \( a < y \), so we will have a “cycle”.
- If this condition is violated, both players will exert infinite effort.
No NE in Pure Strategies (Continued)

- For example, suppose that $a = 5, \alpha = 0.6, \beta = 1.1$.
- Note that $1 + \beta - 3\alpha \beta = 0.12 > 0$. 

![Graph showing payoffs against effort. The graph has a red line labeled 'Free ride' and a dashed black line labeled 'Work slightly harder than the opponent'.]
Symmetric Mixed Strategy Equilibrium of the Stage Game

- Assume that $1 + \beta - 3\alpha\beta > 0$ and focus on the symmetric mixed strategy Nash equilibrium of this game.
- Both players randomize according to a cdf $F(x)$ with support $[l, h]$.
- If $j$ uses this strategy and $i$ chooses $x_i$, $i$’s expected payoff will be
  \[
  u_i = \int_{l}^{x_i} \beta \alpha (x_i + x_j) dF(x_j) + \int_{x_i}^{h} \beta (1 - \alpha) (x_i + x_j) dF(x_j) - x_i.
  \]
- To randomize, $i$ must be indifferent between any $x_i$ in $[l, h]$.
- Thus, $F(x)$ must be such that $\partial u_i / \partial x_i = 0$ for any $x_i$.
- Differentiating w.r.t. $x_i$ and using the Leibnitz rule yields
  \[
  F(x_i) + 2x_i F'(x_i) - R = 0,
  \]
  where
  \[
  R = \frac{1 - \beta (1 - \alpha)}{\beta (2\alpha - 1)} > 1.
  \]
Solving the Differential Equation

- The solution to this differential equation is $F(x_i) = R + C/\sqrt{x_i}$, where $C$ is the constant of integration.

- We also need to determine the support of $F$.

- The constant $C$ must satisfy $F(l) = R + C/\sqrt{l} = 0$.

- Solving for $C$ yields $C = -R\sqrt{l}$.

- The upper bound of the support satisfies $F(h) = R(1 - \sqrt{l}/\sqrt{h}) = 1$.

- Solving for $h$ gives us $h = l \left(\frac{R}{R-1}\right)^2$.

- So the cdf $F$ of each player’s strategy is $F(x) = R \left(1 - \frac{\sqrt{l}}{\sqrt{x}}\right)$.

- The corresponding pdf is $f(x) = F'(x) = \frac{\sqrt{l}Rx^{-1.5}}{2}$. 
The expected effort of player $i$ is $\bar{x}_i = \frac{l \left( \frac{R}{R-1} \right)^2}{\int l x_i \frac{\sqrt{lR} x_i^{1.5}}{2} dx_i} = l \left( \frac{R}{R-1} \right)$. 

If $j$ sticks to this strategy, $i$ will be indifferent between any $x_i$ in $[l, h]$. To get $i$'s expected payoff, we can set $x_i = l$ and substitute this in $u_i$: 

$$\int_{l}^{h} \beta (1 - \alpha) (l + x_j) dF(x_j) - l = \beta (1 - \alpha) l \left( \frac{R}{R-1} \right) - l [1 - \beta (1 - \alpha)].$$

But if $i$ chooses $a$ and $a < l$, he will get a higher expected payoff: 

$$\int_{l}^{h} \beta (1 - \alpha) (a + x_j) dF(x_j) - a = \beta (1 - \alpha) l \left( \frac{R}{R-1} \right) - a [1 - \beta (1 - \alpha)].$$

Therefore, it must be the case that $l = a$!
To recap, the equilibrium mixed strategy cdf is

\[ F(x) = R \left( 1 - \frac{\sqrt{a}}{\sqrt{x}} \right), \]

where \( R = \frac{1 - \beta(1 - \alpha)}{\beta(2\alpha - 1)} \).

The strategy support is \([a, a \left( \frac{R}{R-1} \right)^2]\).

Each player’s expected effort is \(a \left( \frac{R}{R-1} \right)\).

Each gets an expected payoff \(\beta(1 - \alpha)a \left( \frac{R}{R-1} \right) - a[1 - \beta(1 - \alpha)]\).

Note that if \(\beta = 1\), we have a zero-sum game.

- In a symmetric equilibrium, each player must expect a zero payoff.
- Indeed, when \(\beta = 1\) we get \(u_i = u_j = 0\).
Illustration

- Let \( a = 5, \alpha = 0.6, \beta = 1.1 \). Expected effort is \( \bar{x} = 8.24 \).
- The pdf is shown below.
The figures illustrate how the game value changes with $\alpha$ and $\beta$. If $\beta < 1$, the game value is negative and decreasing in $\alpha$. 

Game value for $\beta = 1.1$. 

Game value for $\alpha = 0.6$. 

Game value for $\beta = 1.1$. 

Game value for $\alpha = 0.6$. 

Illustration (Continued)
Dynamic Game Setting

- Next I consider a dynamic version of this game.
- In each period, the players play the stage game described above.
- Their period-$t$ objective is to maximize the discounted sum of expected instantaneous payoffs

$$
U_1^t = \sum_{\tau=t}^{\infty} \delta^{(\tau-t)} E_t(u_1^\tau), \quad U_2^t = \sum_{\tau=t}^{\infty} \delta^{(\tau-t)} E_t(u_2^\tau).
$$

- The state variable is the minimum effort level $a^t$ (“reputation”).
- Assume that it evolves according to $a^{t+1} = \mu + \eta(x_i^t + x_j^t)$.
- Again, using arguments similar to those in the one-shot game, we can show that a pure strategy equilibrium does not exist.
- Focus on the symmetric equilibrium in Markovian mixed strategies - the cdf is conditional on the state:

$$
F^t = F(\cdot|a^t).
$$

- Note that such an equilibrium will be subgame-perfect.
Analysis of the Dynamic Game

- Now players have two types of incentive to exert effort: static (to win the current stage game) and dynamic (to affect future effort).
- Let $U_i(a^{t+1})$ be the continuation value of the game for player $i$.
  - That is, $U_i(a^{t+1})$ is sum of $i$’s expected discounted payoffs from $t + 1$ onward, where the expectation is taken at the beginning of period $t + 1$ (i.e., once $a^{t+1}$ has been realized).
- We conjecture that $U_i(a^{t+1})$ is linear in $a^{t+1}$: $U_i(a^{t+1}) = r + sa^{t+1}$.
- Therefore, $i$’s expected lifetime payoff at time $t$ is

$$
\int_{x_i^t}^{h^t} \beta \alpha (x_i^t + x_j^t) dF(x_j^t|a^t) + \int_{x_i^t}^{h^t} \beta (1 - \alpha) (x_i^t + x_j^t) dF(x_j^t|a^t) - x_i^t \\
+ \delta E_t(U_i(a^{t+1}))
$$

where $E_t(U_i(a^{t+1})) = \int_{x_i^t}^{h^t} [r + s(\mu + \eta (x_i^t + x_j^t))] dF(x_j^t|a^t)$. 
Again, $i$ must be indifferent between any $x^t_i$ in $[l^t, h^t]$.
Therefore, $F(\cdot | a^t)$ must be such that the derivative of the expression for the expected lifetime payoff is 0 for any $x^t_i$.
Differentiation w.r.t. $x^t_i$ yields the following differential equation:

\[ F(x^t | a^t) + 2x^t_i F'(x^t | a^t) - \tilde{R} = 0, \]

where $\tilde{R}$ is now given by

\[ \tilde{R} = \frac{1 - \beta(1 - \alpha) - \delta \eta}{\beta(2\alpha - 1)}. \]

Using similar arguments to those in the static game, it can be shown that the period-$t$ strategy support is $[a^t, a^t \left( \frac{\tilde{R}}{\tilde{R} - 1} \right)^2]$.

As before, solving the differential equation yields

\[ F(x^t | a^t) = \tilde{R} \left( 1 - \frac{\sqrt{a^t}}{\sqrt{x^t}} \right). \]
Now we pin down the coefficients $r, s$ of the continuation payoff.

We can define $U_i(a)$ recursively.

Let $h^t = a^t \left( \frac{\tilde{R}}{\tilde{R} - 1} \right)^2$. For any $x_i^t$ in $[a^t, h^t]$, we have

$$r + sa^t = \int_{a^t}^{x_i^t} \beta \alpha (x_i^t + x_j^t) dF(x_j^t|a^t) + \int_{x_i^t}^{h^t} \beta (1 - \alpha) (x_i^t + x_j^t) dF(x_j^t|a^t)$$

$$- x_i^t + \delta \int_{a^t}^{h^t} [r + s (\mu + \eta (x_i^t + x_j^t))] dF(x_j^t|a^t).$$

Setting $x_i^t = a^t$ gives us

$$r + sa^t = \beta (1 - \alpha) \left[ a^t + a^t \tilde{R} / (\tilde{R} - 1) \right] - a^t$$

$$+ \delta \left\{ r + s \left[ \mu + \eta \left( a^t + a^t \tilde{R} / (\tilde{R} - 1) \right) \right] \right\}.$$
The method of undetermined coefficients yields equations for $r$ and $s$. The parameter $s$ must satisfy

$$ s = \beta (1 - \alpha) \left[ 1 + \frac{\tilde{R}}{\tilde{R} - 1} \right] - 1 + \delta s \eta \left[ 1 + \frac{\tilde{R}}{\tilde{R} - 1} \right]. $$

Define $A = 1 - \beta (1 - \alpha)$, $B = \beta (2 \alpha - 1)$, $D = \beta (1 - \alpha)$. Also, let $Z$ be the discriminant of the quadratic equation for $s$:

$$ Z = [A - B + \delta \eta (2D - 2A + B - 1)]^2 - 4\delta \eta (2\delta \eta - 1) [A - B - D(2A - B)]. $$

There are 2 solutions for $s$, but we focus on this one:

$$ s = \frac{-[A - B + \delta \eta (2D - 2A + B - 1)] + \sqrt{Z}}{2\delta \eta (-1 + 2\delta \eta)}. $$
Also, the coefficient $r$ must solve $r = \delta r + s\mu$. Therefore,

$$r = \frac{\delta s\mu}{1 - \delta}.$$ 

Again, this is a zero-sum game when $\beta = 1$, so we must get $U_i(a^t) = 0$.

With the above expression for $s$, we do obtain $r = s = 0$.

- The other solution of the equation for $s$ does not satisfy this property.

A necessary condition for the existence a mixed strategy equilibrium is $Z \geq 0$.

This condition is satisfied if $\delta$ and $\beta$ are not too big.

Otherwise, players will want to exert infinite effort.
To recap, the Markovian symmetric mixed strategy is characterized by

$$F(x^t | a^t) = \left[ \frac{1 - \beta(1 - \alpha) - \delta s \eta}{\beta(2\alpha - 1)} \right] \left(1 - \frac{\sqrt{a^t}}{\sqrt{x^t}}\right),$$

The support of the strategy is $[a^t, a^t \left( \frac{1 - \beta(1 - \alpha) - \delta s \eta}{1 - \beta(1 - \alpha) - \delta s \eta - \beta(2\alpha - 1)} \right)^2].$

Each player's expected effort is

$$\bar{x}^t = a^t \left( \frac{1 - \beta(1 - \alpha) - \delta s \eta}{1 - \beta(1 - \alpha) - \delta s \eta - \beta(2\alpha - 1)} \right).$$

Given $a^t$, the expected value of the game is

$$U(a^t) = -\left[ A - B + \delta \eta(2D - 2A + B - 1) \right] + \sqrt{Z} \left( \frac{\delta \mu}{1 - \delta} + a^t \right).$$

The variables $r, s, A, B, D, Z$ are as defined above.
Long-Run Forecasting

- Suppose that we want to make a forecast about the effort levels and the value of the game many periods from now.
- Such a forecast should be independent of the current state, and so can be used as a criterion to compare different games.
- The period-$t$ expected value of the state in period $t + \tau$, $a^{t+\tau}$, is

$$E_t a^{t+\tau} = \left( \frac{2\eta \tilde{R}}{\tilde{R} - 1} \right)^\tau a^t + \sum_{\zeta=0}^{\tau-1} \mu \left( \frac{2\eta \tilde{R}}{\tilde{R} - 1} \right)^\zeta.$$

- Suppose that $2\eta \tilde{R} / (\tilde{R} - 1) < 1$. Then we have

$$\lim_{\tau \to \infty} \left( \frac{2\eta \tilde{R}}{\tilde{R} - 1} \right)^\tau a^t = 0.$$

- Thus, for $\tau \to \infty$, we get

$$E_t a^{t+\tau} = \tilde{a} \approx \frac{\mu(\tilde{R} - 1)}{\tilde{R}(1 - 2\eta) - 1}.$$
In the long run, a player’s effort will be somewhere in the interval

\[
\left[ \frac{\mu}{1 - 2\eta'} \frac{\mu(\tilde{R} - 1)^2}{(\tilde{R} - 1)^2 - 2\eta \tilde{R}^2} \right]
\]

The long-run forecast for each player’s effort is

\[
\frac{\mu(\tilde{R} - 1)}{\tilde{R}(1 - 2\eta) - 1} \left( \frac{1 - \beta(1 - \alpha) - \delta s\eta}{1 - \beta(1 - \alpha) - \delta s\eta - \beta(2\alpha - 1)} \right).
\]

The long-term forecast for the value of this game to a player is

\[
\sqrt{\mathcal{Z}} - \left[ A - B + \delta \eta (2D - 2A + B - 1) \right] \left( \frac{\delta \mu}{1 - \delta} + \frac{\mu(\tilde{R} - 1)}{\tilde{R}(1 - 2\eta) - 1} \right).
\]

Note that when \( \delta \to 0 \), our long-term forecasts converge to the predictions for a static game in which \( a = a \).
Numerical Example: Dynamic Incentives

- We set parameter values $\eta = 0.2, \mu = 5, \delta = 0.9, \beta = 1.1$.

- The long-term forecast for the value of the game is increasing in $\alpha$.
- If $\beta < 1$, the game value is negative and decreasing in $\alpha$. 
Extension: The Winner Determines the State

- Suppose that the winner of the stage game determines the state:
  \[ a^{t+1} = \mu + \eta \max\{x_1^t, x_2^t\}. \]

- Conjecture that \( U_i^{t+1} = r + sa^{t+1} \). Mr. \( i \)'s expected lifetime payoff is
  \[ \int_{l^t} x_i^t \frac{\beta \alpha (x_i^t + x_j^t) dF(x_j^t|a^t)}{x_i^t} + \int_{x_i^t}^{h^t} \beta (1 - \alpha) (x_i^t + x_j^t) dF(x_j^t|a^t) - x_i^t \]
  \[ + \delta \int_{l^t} [r + s(\mu + \eta x_i^t)] dF(x_j^t|a^t) + \delta \int_{x_i^t}^{h^t} [r + s(\mu + \eta x_j^t)] dF(x_j^t|a^t). \]

- Differentiation yields \( F(x^t|a^t) + Gx^tF'(x^t|a^t) - \hat{R} = 0 \), where
  \[ G = \frac{2\beta (2\alpha - 1)}{\beta (2\alpha - 1) + \delta \eta}, \quad \hat{R} = \frac{1 - \beta (1 - \alpha)}{\beta (2\alpha - 1) + \delta \eta}. \]
The solution to this differential equation is

\[ F(x^t|a^t) = \hat{R} \left( 1 - \left( \frac{a^t}{x^t} \right)^{\frac{1}{G}} \right). \]

Now the strategy support is \([a^t, a^t (\frac{R}{R-1})^G]\).

The expected effort is

\[ \bar{x}_i^t = \frac{\hat{R}a^t}{(G-1)} \left( \frac{\hat{R}^{G-1}}{((\hat{R} - 1)^{G-1} - 1) \right). \]

Next we determine the coefficient’s \(r, s\) of the continuation payoff.

Again, we can apply the method of undetermined coefficients.
Setting $x^t = a^t$ allows us to write the following recursive equation:

$$r + sa^t = \frac{\beta(1 - \alpha)\hat{R}a^t}{(G - 1)} \left( \frac{\hat{R}^{G-1}}{\left(\hat{R} - 1\right)^{G-1}} - 1 \right) - a^t[1 - \beta(1 - \alpha)]$$

$$+ \delta \left\{ r + s \left[ \mu + \eta \frac{\hat{R}a^t}{(G - 1)} \left( \frac{\hat{R}^{G-1}}{\left(\hat{R} - 1\right)^{G-1}} - 1 \right) \right] \right\}.$$

Thus, $s$ must satisfy

$$s = \frac{\beta(1 - \alpha)\hat{R}}{(G - 1)} \left( \frac{\hat{R}^{G-1}}{\left(\hat{R} - 1\right)^{G-1}} - 1 \right) \left[ \beta(1 - \alpha) + \delta s \eta \right] - [1 - \beta(1 - \alpha)].$$

The equation for $s$ can be solved numerically.

- Note that $s$ appears in $\hat{R}$ and $G$.

- Also, $r$ is given by

$$r = \frac{\delta s \mu}{1 - \delta}.$$