Writing models as matrix exponentials

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Writing the model - ODEs

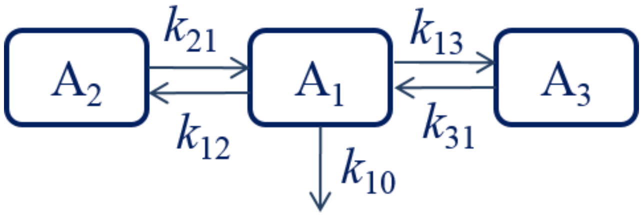
$$\frac{dA_1}{dt} = -(k_{12} + k_{13} + k_{10})A_1 + k_{21}A_2 + k_{31}A_3 \qquad A_{1(0)}=\mathrm{Dose}$$

$$\frac{dA_2}{dt} = k_{12}A_1 - k_{21}A_2 \qquad A_{2(0)}=0$$

$$\frac{dA_3}{dt} = k_{13}A_1 - k_{31}A_3 \qquad A_{3(0)}=0$$

Most complicated PK models are written as ODEs. Note here that this simple 3 compartment model can be written as an algebraic closed form solution. But it also illustrates the idea.

Re-writing an ODE as a matrix of rate constants



- How mass transfer is written

$$K = \begin{bmatrix} all\ losses\ from\ A_1 & transfer\ A_2 \rightarrow A_1 & transfer\ A_3 \rightarrow A_1 \\ transfer\ A_1 \rightarrow A_2 & all\ losses\ from\ A_2 & transfer\ A_3 \rightarrow A_2 \\ transfer\ A_1 \rightarrow A_3 & transfer\ A_2 \rightarrow A_3 & all\ losses\ from\ A_3 \end{bmatrix}$$

- Mass transfer matrix of rate constants

$$K = \begin{bmatrix} -(k_{12} + k_{13} + k_{10}) & k_{21} & k_{31} \\ k_{12} & -k_{21} & 0 \\ k_{13} & 0 & -k_{31} \end{bmatrix}$$

Writing the model – as matrix exponential

$$\begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \end{bmatrix} = Pe^{t\Lambda}P^{-1}\begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}, C(t) = A_1(t)/V_1$$

- Where

P = eigenvector of K
Λ = eigenvalues of K

- MATLAB/Freemat command

$$[P, \Lambda] = eig(K)$$

As long as the model is linear and time-invariant then any PK or PD model can be written as a matrix exponential solution. This solution is exact and is much quicker than using ODE solving.

- See also the Matrix exponential download.